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LONGITUDINAL INSTABILITIES OF RELATIVISTIC BEAMS IN AXIALLY SYMMETRIC MAGNETIC FIELDS

R. W. Landau

SIGNAL CORPS CONTRACT Nr. DA36-039-SC-87242
ARPA Order Nr. 112-61 Project Code Nr. 7600

United States Army
Signal Research and Development Laboratory
Fort Monmouth, New Jersey

STEVENS INSTITUTE OF TECHNOLOGY
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ABSTRACT

In this paper, a canonical formalism has been developed for the description of the negative mass instability (N.M.I.) and longitudinal oscillations of relativistic beams. This formalism has been applied to ascertain the stabilizing effect of betatron oscillations, and to determine the dispersion relation governing counterstreaming ions and relativistic electrons. The results show that only the spread in p , the canonical angular momentum of the particles, contributes to stability. The N.M.I. equation for two streams is the same as though each were separately present; and the dispersion relation for longitudinal oscillations of beams in a magnetic field is given by the N.M.I dispersion relation, and not by the dispersion relation for longitudinal oscillations of collinear beams. Moreover, the dispersion relation for longitudinal oscillations of thin collinear beams differs from the usual equation by a non-trivial factor.

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NOTE

The equations in the four chapters are labelled by number, e.g., eq. 2-34, the 34th equation in Chapter II.

The equations in the Appendices are labelled by letter, e.g., eq. D-12, the 12th equation in Appendix IV.

Within the chapter, eq. 2-34 is referred to simply as eq. 34.

Within the appendix, eq. D-12 is referred to simply as eq. 12.

The symbols used and some words are explained and defined in the Definitions and Symbols Section (p.123)

CHAPTER I - INTRODUCTION

Sec. 1.1 - The Description of the N.M.I.

The N.M.I. causes azimuthal clumping of intense beams in any device with a magnetic field of axial symmetry that provides radial and axial focusing, e.g., synchrotron, betatron, and mirror machine (e.g., DCX). (See Fig. 1 .) In the unstable regime this clumping grows so that eventually transverse space charge forces exceed the focusing forces and part of the beam is lost. (A crude analogy is the loss of water in a shallow circular trough, when waves are set up.) In the stable regime, the clumps travel with the beam and move very slowly relative to it. Thus l equally spaced clumps in a beam moving at cyclotron frequency $\dot{\theta}_0$, will give rise to a charge variation at one point of frequency $\omega = l\dot{\theta}_0$.

A simple explanation of the cause of the instability is the following. Consider an azimuthally uniform distribution of cold particles in a beam. Making a small sinusoidal perturbation in the beam density effects a sinusoidal electric potential, rotating with the beam. Those particles ahead of the potential bump will be speeded up and those behind it will be slowed down. Thus one would expect the bump to evanesce. However, those particles which were speeded up move outward radially due to the centrifugal force. The amount of radial motion depends on the magnetic field shape. For weak focusing machines ($0 < n < 1$), the radial excursion is large enough to overcompensate the increase in linear velocity so that the angular velocity decreases. On the other hand, those particles that are slowed down in linear velocity, move inward radially and thus speed up in angular velocity so that the net effect is that the particles tend to move toward the angular position of the potential bump and the perturbation grows. Since the angular acceleration is opposite to the force, the effect is as though the particles have negative effective mass.

As the radial focusing increases in strength, the radial excursion becomes relatively smaller so that an increase in linear velocity causes an increase in angular velocity and the effective mass becomes positive. This is the situation, for example, in strong focusing machines (below transition energy) where the N.M.I. will not occur.

Gravitational forces are weak, in the above sense, so that particles have an effective negative mass. Such a situation exists in Saturn's rings. There, however, the forces between two particles are attractive so that the negative mass prevents clumping and the ring system is stable. This was first pointed out by Maxwell.[1]

The cause of the instability may also be seen by examining normal synchrotron operation. During the acceleration cycle of a synchrotron, an R. F. field is applied across a gap. If we assume that the gap is so small that the time change of the gap field is negligible while it is being traversed by the particle, then the energy gained by the particle is dependent only on its phase relative to the R. F. field. In this case if the R. F. peak gap potential is eV, then the potential may be replaced by an equivalent, continuous rotating potential over the whole path of the particle given by $\frac{eV}{2\pi} \cos(\theta - \omega_{R.F.} t)$.[2] The energy gained in one cycle of the particle is equal in both cases. Analysis of the phase motion shows that there is a point of phase stability near the peak of the rotating field. The beam particle density is therefore a maximum at the same point. If the magnetic field is stationary then the stable point is at the peak. Suppose now that there is no rotating external electric field but a rotating internal field caused by θ perturbations in the beam density. Since the stable phase point is at the field maximum, the particle density at the maximum will grow, which will make the field still stronger, resulting in beam clumping.

Sec. 1.2 - Background

This work was motivated by the conjecture that the N.M.I. is the effect which most severely limits the maximum currents allowed in a plasma betatron. For a plasma at an initial temperature of 3 e.v. the linearized N.M.I. theory predicts stability at a neutral beam density where the current is only $\lesssim \frac{1}{10}$ amp of relativistic electrons (see eq. 2-47).

The plasma betatron is a device which accelerates a neutralized beam of positive ions and electrons so that the space charge limitations of ordinary machines do not apply. Examination of the equilibrium conditions, by including the effect of the self-magnetic field as done by Schmidt,[3] gives the limit $\frac{\nu g}{\gamma} < \epsilon_n$, which permits 1000 amperes.

Instabilities were first discussed by Budker,[4] the originator of the scheme of the acceleration of a neutralized beam. Two of these instabilities, the two-stream longitudinal and the transverse (sinuous) are also discussed by Finkelstein and Sturrock,[5] who find stability criteria much less restrictive than required for the N.M.I.

Another instability which might severely limit the maximum beam current was noted by Rosenbluth.[6] It arises when a beam of particles passes through a background resistive plasma. Since in principle, this effect may be eliminated in a plasma betatron by careful design, we will not consider it further.

Harrison[7] has pointed out that the two-stream longitudinal instability severely restricts the maximum currents in a non-relativistic electron beam. As already pointed out in a paper by Finkelstein and Sturrock,[5] hereafter to be referred to as F-S, for relativistic beams, $\gamma < 2000$ for stability, however for slower beams as shown by Harrison the stability requirement is $\frac{V_-}{c} > \sqrt{\nu_- g}$, for cold electron and ion beams. (ν_- is the lineal stream density multiplied by the classical electron radius; V_- is the electron stream velocity, the ion velocities being small; g is a logarithmic geometrical factor of order unity.) The theory of the N.M.I. shows that even if this inequality is satisfied each beam must be hot enough so that $\frac{\Delta V_-}{c} > \sqrt{\frac{\nu_-}{\alpha} g}$, $\frac{\Delta V_+}{c} > \sqrt{\frac{\nu_+}{\alpha} g}$. (Since $\sqrt{\alpha} \sim 1$ for usual n values, and g is a log term, this inequality is independent of the device considered.) We see therefore that the N.M.I., discovered independently by Nielsen, Sessler and Symon,[8] and Kolomenskii and Lebedev,[9] restricts the maximum currents in a plasma betatron more severely than the other effects.

Plasma betatrons have been built by Budker and Naumov[10] and by workers at CERN[11] with the result that maximum currents were 10 amperes, much below the design value of these machines. The limitation is possibly due to the N.M.I. A plasma betatron is also under study at Stevens Institute of Technology.[12]

This instability is of broader interest, because, as we shall show, thin beams in mirror magnetic fields where $0 < n < 1$, will also be subject to it, e.g. in the DCX[13] machine, where it may be the cause of the observed frequencies, as also pointed out by Fowler.[14] The Astron,[15] containing a beam in a mirror field may also be subject to the N.M.I. Samoilov[16] and Seidl[17] have observed particle bunching in betatrons and attribute this to the N.M.I. However, sufficiently detailed measurements have not been made to verify this conjecture. These authors also suggest that the N.M.I. is the major cause of capture of particles into stable orbits in betatrons.

Sec. 1.3 - Outline

We have derived the N.M.I. equations using a canonical formalism and the relativistic Hamiltonian. This procedure allows one to include additional effects easily. With the resultant dispersion relations, we derive a necessary and sufficient criterion for stability, which is simple only for single humped distributions. The stability criterion may be given explicitly for Maxwellian distributions, and is similar to the result obtained for rectangular pulse distribution functions by other authors[8], [9] and hence justifies the use of pulse functions. Our dispersion relation differs somewhat from earlier results so that stable distributions exhibit damped oscillations. This effect is shown explicitly for a resonance distribution function.

Kolomenskii and Lebedev[9] have obtained similar stability criteria for a resonance function but have not worked out the damped situation. This damping is mathematically analogous to Landau damping[13] in infinite plasmas. Our results exhibit no damping for pulse distribution functions in agreement with the results of other authors.

Previous works of other authors have dealt with circulating beams enclosed within conducting boundaries. This paper considers unshielded beams so that every part of the beam sees every other part. We find that for small wave numbers of the perturbation, in the relativistic domain, the beam will be stable even if it is cold. (If $\frac{1}{\gamma_g^2} \equiv \frac{1}{\gamma^2} - \frac{v^2}{c^2}\frac{\delta g_l}{g}$ is negative then there is stability, where $\delta g_l \approx \frac{1}{l^2}$, $l = 1, 2 \ldots$ and $g \approx 5$ for typical cases.)

Next we consider the effect of betatron oscillations on the N.M.I. and treat separately the axial (z) and radial (r) oscillations. We find that these have a very slight effect on stability and therefore only the spread of p_θ contributes to the stability. Our calculations also show that the growth rate of the instability slows down as it approaches the radial betatron oscillation frequency where the equations break down.

Finally we generalize our equations to find the dispersion relation under the N.M.I. for two streams, counterstreaming ions and relativistic electrons. We find that the stability criteria are almost the same as though each beam were present by itself. The difference is that when $n > \frac{1}{2}$ and $1 \gg \frac{\nu m_- g}{m_+} > \frac{1}{\gamma^2}$, the electron modes are stable, even for a cold electron beam. The ion modes, however, are still unstable.

Next it is shown that the equation for longitudinal oscillations of collinear streams is valid for circular streams, only when $\frac{1}{1-h} \ll \frac{1}{\gamma^2}$, which is not generally true. (It is true only in strong-focusing machines below transition.) The equation for longitudinal oscillations may therefore be derived from the two-stream N. M. I. equations by letting $\frac{1}{1-h} \rightarrow 0$. This equation, valid only for small ν because the N. M. I equations are restricted to small ν , differs from the F-S[5] longitudinal equation. Going back to the basic equations, an equation for the longitudinal oscillations, valid for all ν , is obtained which gives stability for even higher currents than found in F-S. To check the validity of our equation, the dispersion relation for two infinitely wide beams is obtained from it. If the ion beam is stationary and cold, the dispersion relation agrees with that found by Bludman et al.[19] If we set the number of ions equal to zero, we find that the resulting dispersion relations may be obtained by a Lorentz transformation from the dispersion relations of both thin and infinite beams of non-relativistic electrons. These results show that our modification of the F-S equation, which consists of a factor $1 - \left(\frac{r_0 \Omega}{2c}\right)^2$, is correct. This equation is similar to eq. 32 (eq. 9 in the abridged translation) of Budker,[4] who has obtained the same stability criterion. (His equations neglect beam temperatures and the ion beam velocity.)

CHAPTER II - ONE STREAM

Here we develop the dispersion relations for a single stream. A constraint equation is found which reduces the Boltzman Equation to a one-dimensional equation. Since the equation for the potential is then given by an integral over one momentum variable, the resulting equations are formally similar to the one-dimensional system first studied by L. D. Landau[18] and later also by Backus.[20] These equations are solved following Jackson[21] while the Nyquist diagram technique of Penrose[22] is used to obtain generalized stability criteria including the stability criterion for a Maxwellian distribution. These equations are then solved exactly for a resonance function and a pulse function. Finally the Boltzman Equation is solved, non-relativistically, by including the (z) axial betatron oscillations and the (r) radial betatron oscillations separately.

Sec. 2.1 - The Constraint Equation

The basic equation for our system is the collisionless Boltzman equation

$$\frac{\partial \Psi}{\partial t} + \dot{\theta}\frac{\partial \Psi}{\partial \theta} + \dot{p}_\theta \frac{\partial \Psi}{\partial p_\theta} + \dot{r}\frac{\partial \Psi}{\partial r} + \dot{p}_r \frac{\partial \Psi}{\partial p_r} + \dot{z}\frac{\partial \Psi}{\partial z} + \dot{p}_z \frac{\partial \Psi}{\partial p_z} = 0 \qquad 2\text{-}1$$

If the coefficients $\dot{q}_i$ and $\dot{p}_i$ are obtained from a relativistic Hamiltonian and Ψ describes particles with the same rest mass, then this equation is relativistically correct. A brief discussion of the relativistic invariance is given by Belyaev and Budker.[23] In the following way they show that Ψ is a Lorentz invariant scalar. The particle flux and density four-vector, j_k is obtained from $\int F_k d^4p = j_k$, an invariant expression. In terms of the four-velocity u_k, $F_k = Fu_k$ and $F(x_k, p_k) = -i\Psi(x_k, p_k)\cdot\delta\left(\sqrt{-(p_k - \frac{e}{c}A_k)^2} - m_o c\right)$. The δ function arises because F describes particles with the same rest mass. Ψ is the distribution function of eq. 1, so that $\int \Psi d^3p = n$, the number of particles per unit volume. Since the δ function is written in an invariant way, Ψ must be invariant because F is invariant. (There is an error in the expression for F and H in their paper.(23) The factor in F should be $-i$, not ic, and H should be multiplied by c.) Because of the δ function, Ψ is a function of only seven variables, the four coordinates and the three momenta. Eq. 1 may be derived from their invariant Boltzman equation by integrating it over dp_4 to eliminate the δ function. Thus eq. 1 is relativistically correct.

Our procedure is to simplify the expressions for the coefficients of eq. 1 and then solve the Boltzman equation by a perturbation procedure. This means writing $\Psi = \Psi_0 + \Psi_1$, where $\Psi_1 \ll \Psi_0$ in some operational sense and Ψ_0 describes the unperturbed configuration which is time independent. If Ψ_1, initially small, has an exponentially increasing time dependence then the system is unstable.

The unperturbed system consists of one specie of particles rotating about an axially symmetric magnetic field. (See Fig. 1). The B_z field falls off slowly with radius near r_o according to $B_z = B_o\left(\frac{r_o}{r}\right)^n$ where $0 < n < 1$ to provide focusing, as explained in Appendix I. The particles occupy a toroidal region of small cross section and form an azimuthally uniform distribution.

The equations for the transverse motion are derived in Appendix I and are given by eqs. A-8, 9 and 10, which describe the usual betatron oscillations. Note that m, the relativistic mass (γm_0, as defined in Appendix I) is constant, since the energies of betatron oscillation are constant. These equations for the transverse motion are valid if

$$\frac{\partial A_\theta^0}{\partial r}, \quad \frac{\partial A_\theta^0}{\partial z}, \quad \frac{\partial \varphi^0}{\partial r}, \quad \frac{\partial \varphi^0}{\partial z} \qquad 2\text{-}2$$

are neglected relative to $\frac{\partial A_B}{\partial r}$, $\frac{\partial A_B}{\partial z}$ in eqs. A-3a, 3c. (The superscript zero refers to quantities due to the unperturbed beam.) This requires that $\frac{\nu}{\gamma^3} \ll \frac{1-n}{2}\left(\frac{v}{c}\,\frac{\rho}{r_0}\right)^2 \equiv \epsilon_e$ for the electron$^{(-)}$ or ion$^{(+)}$ stream. It is also required that A_r, A_z and A be negligible. In fact $A_r = A_z = 0$ because of the symmetry of the particle motion. Finally, as Schmidt[(3)] has pointed out, the self-field term, $\frac{\partial A_\theta^0}{\partial r}$, causes a radial shift in equilibrium orbit which is negligible if $\frac{\nu g}{\gamma} \ll \epsilon_M$ in which case A_θ^0 is also negligible.

The Schmidt criterion may be derived in the following manner. $B_z^0 = 0$ at the center of the current torus, i.e., near $r = r_0$, $z = 0$, if the current is distributed uniformly over the cross section. The relation $B_z^0 = \frac{\partial A_\theta^0}{\partial r} + \frac{A_\theta^0}{r} = 0$ then gives $\frac{\partial A_\theta^0}{\partial r} = -\frac{A_\theta^0}{r}$. From Appendix I we see that the radial motion of the particles is determined by a vector potential A_θ. The beam center is at the bottom of the well defined by the total vector potential, the external plus the self-field. See Fig. 2. The location of the bottom is given by the solution of $\frac{\partial A_\theta}{\partial r} = \frac{\partial A_B}{\partial r} + \frac{\partial A_\theta^0}{\partial r} = 0$ or $\frac{\partial A_B}{\partial r} = \frac{A_\theta^0}{r}$ using $B_z = 0$. To find the value of A^0_θ at the center of the beam, near $r = r_0$, we use eqs. A-5b, C-17 and C-20 to obtain $A^0_\theta = \frac{e\hat{g}_0}{r_0}\frac{N}{2\pi}\frac{v_\theta}{c} = \nu\hat{g}_0 \frac{m_0 c}{e} v_\theta$. By eq. A-2b, $v_\theta = -\frac{eA_\theta}{mc} = \frac{-e}{\gamma m_0 c}(A^0_\theta + A_B)$. Substituting this value of v_θ and solving for A^0_θ gives $A^0_\theta = \dfrac{\frac{-\nu\hat{g}_0 A_B}{\gamma}}{1+\frac{\nu\hat{g}_0}{\gamma}}$. By eq. A-4, $\frac{\partial A_B}{\partial r} = (1 - n)\frac{r-r_0}{r_0^2}A_{B0}$. Thus using the equation for the location of the bottom of the well $\frac{\partial A_B}{\partial r} = \frac{A^0_\theta}{r}$, we obtain finally $-\frac{\nu\hat{g}_0}{\gamma r}\frac{A_B}{1+\frac{\nu\hat{g}_0}{\gamma}} = (1-n)\frac{r-r_0}{r_0^2}A_{B0}$. Since $\frac{\nu\hat{g}_0}{\gamma} \ll 1$, and $r \approx r_0$, this expression reduces to

$$\frac{\nu\hat{g}_0}{\gamma} = -(1-n)\frac{(r-r_0)}{r_0}$$

The maximum of the R.H.S. of this equation is ϵ_M , which gives us Schmidt's criterion. Note that the current loop moves radially inward in the Betatron field, contrary to a free current loop which, as is well-known, expands.

The unperturbed, zero-order azimuthally symmetric distribution describes particles with a spread in p_θ values and a range of betatron oscillation amplitudes.

The variables p_θ , $r - r_0$, and z are considered first-order small. Quadratic terms in these quantities will be neglected. As a result of the perturbation which causes azimuthal fields, p_θ is no longer constant for each particle but changes slowly with time (see eq. A-3b) and, therefore, m will too. There will now be terms due to A'_r, A'_θ and φ', in the expression for $\dot{p}_z$, and hence additional terms in eq. A-9. A_z remains zero because the motion is symmetric about the z = 0 plane. A'_r may also be neglected for thin beams because the radial phase velocity due to the N.M.I. is always much slower than c.

The terms $\frac{\partial A'_\theta}{\partial r}$, $\frac{\partial A'_\theta}{\partial z}$, $\frac{\partial \varphi'}{\partial r}$, $\frac{\partial \varphi'}{\partial z}$ give the effect of the perturbed transverse space charge. It appears plausible that under the conditions that the zero order transverse space charge effects may be neglected relative to the focusing betatron field, that the perturbed terms may also be neglected. It is possible in fact to show this non-relativistically with the formalism of Sec. 2.5. This suggests that if $\frac{\nu g}{\gamma^3} \ll \epsilon_e$ these terms may be neglected.

With the above assumptions, we obtain from eqs. A-8 and A-9 that $p_z = m\dot{z}$ and $\dot{p}_z = -\frac{bz}{m}$. Since m must be constant for particle motion in a static magnetic field, these equations give $m\ddot{z} = -\frac{bz}{m}$ or $p_z^2 + bz^2 =$ constant. This means that the energy of the z betatron oscillations is constant. Thus if Ψ is a function of z and p_z only through $p_z^2 + bz^2$, i.e., $\Psi = \Psi(r, p, \theta, p_\theta, t, p_z^2 + bz^2)$, the two z terms in the Boltzman equation add to zero as may be verified by substitution. The coefficients of the other terms do not contain z or p_z to first order so that we may integrate the Boltzman equation over $dz\,dp_z$, and writing $\Psi' = \int \Psi dz\,dp_z$ obtain,

$$\frac{\partial \Psi'}{\partial t} + \dot{\theta}\frac{\partial \Psi'}{\partial \theta} + \dot{p}_\theta \frac{\partial \Psi'}{\partial p_\theta} + \dot{p}_r \frac{\partial \Psi'}{\partial p_r} + \dot{r}\frac{\partial \Psi'}{\partial r} = 0 \qquad \text{2-3}$$

We have thereby reduced the equation to a two-dimensional one.

Next we deduce a constraint equation linking r and p_θ, through which the problem is reduced to only one dimension. The equation of motion in the r direction is, by eq. A-10 (again neglecting the transverse space charge forces and A'_r),

$$\frac{d}{dt} m\dot{r} = -(-V_0)\frac{\partial}{\partial r}\left(-\frac{p_\theta}{r} + \frac{e}{c}A_\theta\right) \qquad \text{2-4a}$$

For m and p_θ constant this equation may be rewritten as

$$\ddot{r} = -\frac{\dot{\theta}_0}{m r_0} p_\theta - \omega_r^2 (r - r_0) \qquad \text{2-4b}$$

where $\omega_r = \sqrt{1-h}\,\dot{\theta}_0 = \sqrt{1-h}\left(\frac{-e B_0}{mc}\right)$. This shows that a particle oscillates about an equilibrium position $r_e = r_0 + \frac{1}{1-h} \frac{c}{e B_0 r_0} p_\theta$ with a frequency ω_r. This equilibrium position will vary with p_θ, but the frequency ω_r remains constant to lowest order. This may be seen more clearly by examining the plot of A_B in Fig. 2. Eq. 4a gives the motion of a particle in the potential well $\frac{p_\theta}{r} - \frac{e}{c} A_\theta$. Clearly the minimum of the total well is shifted according to the value of p_θ. (This is seen by adding the curve $(-\frac{c}{e} p_\theta)\frac{1}{r}$ to A_B in Fig. 2).

Under the influence of azimuthally varying electric fields, p_θ will change, as appears from eq. A-2b. These fields will occur as a result of the N.M.I. Suppose now that the p_θ variation of a particle is very slow. Then if the particle is initially at the bottom of the total well, it will stay very near the bottom and follow the shifts in p_θ. This may be shown easily by writing the steady state solution of eq. 4 with the initial condition that the particle is resting at the bottom of the well, i.e., $\dot{r} = \ddot{r} = 0$ at t = 0 and the assumption that $p_\theta = P \cos \omega_N t$. This solution is

$$r - r_0 = \frac{\dot{\theta}_0}{m r_0} \frac{P}{\omega_N^2 - \omega_r^2} \left[\cos \omega_N t - \frac{\omega_N^2}{\omega_r^2} \cos \omega_r t \right] \qquad \text{2-5}$$

If now the p_θ oscillation is so slow that $\omega_N^2 \ll \omega_r^2$, then the second term on the R.H.S. of eq. 5 may be neglected and we find

$$r - r_0 = -\frac{1}{1-h} \frac{c}{e B_0 r_0} p_\theta \qquad \text{2-6}$$

i.e., the particle follows the bottom of the well. This may also be seen from eq. 4b as $\ddot{r}$ is now much less than either term on the R.H.S. of the equation. Setting $\ddot{r} = 0$ gives the same eq. 6. This is the desired constraint equation. This means that if the particle is initially at the bottom of the well, with nearly zero amplitude betatron oscillations, then if the bottom of the well shifts slowly enough, the particle will follow the bottom without any fast betatron oscillations being excited. Their amplitude by eq. 5 is only $\frac{\omega_N^2}{\omega_r^2}$ the amplitude of the slow motion.

Due to the p_θ changes and the azimuthal field, the relativistic mass m will change with time, so that strictly a term $\frac{dm}{dt}$ should be included on the L.H.S. of eq. 3. This term is however of order p_θ^2 and is neglected as we keep only terms $\sim p_\theta$.

The restriction $\omega_N^2 \ll \omega_r^2$, naturally places restrictions on the solution of the Boltzman equation describing the N.M.I. which restrictions we now derive. We shall assume below that $p_\theta \sim e^{il\theta - i\Omega t}$. The particles travel at an average velocity $\dot{\theta}_0$, so that $\theta = \dot{\theta}_0 t$. Hence, the time variation of p_θ for a particle, is $p_\theta \sim e^{i(l\dot{\theta}_0 - \Omega)t}$, and the frequency of oscillation of p_θ of a particle, is $l\dot{\theta}_0 - \Omega$. Thus the above condition, that the constraint equation be valid is

$$\left|\frac{\omega_N}{\omega_r}\right|^2 \ll 1 \qquad \text{or} \qquad \left|\frac{\Omega - l\dot{\theta}_0}{\sqrt{1-n}\;\dot{\theta}_0}\right| \ll 1 \qquad 2\text{-}7$$

which may be verified to be consistent with the dispersion relation obtained below.

We shall now show how the constraint equation may be used to reduce the Boltzman equation by one more dimension. Consider the quantity

$$x \equiv r - r_e = r - r_o + \frac{1}{1-n} \frac{c}{e B_o r_o} p_\theta$$

which measures the deviation of the particle from the bottom of the well. We shall make a transformation of Ψ' from the variables θ, t, p_θ, p_r, r to θ, t, p_θ, p_r, x. Keeping in mind the fact that $x = f(r, p_\theta)$ we may write the last three terms of the Boltzman equation, eq. 3, as

$$\dot{p}_\theta \frac{\partial \Psi'}{\partial p_\theta} + \dot{p}_r \frac{\partial \Psi'}{\partial p_r} + \dot{r} \frac{\partial \Psi'}{\partial r} = \dot{p}_\theta \left(\frac{\partial \Psi_x'}{\partial p_\theta} + \frac{\partial x}{\partial p_\theta} \frac{\partial \Psi_x'}{\partial x} \right) + \dot{p}_r \frac{\partial \Psi_x'}{\partial p_r} + \dot{r} \frac{\partial \Psi_x'}{\partial x}$$

where $\Psi_x' = \Psi_x'(\theta, t, p_\theta, p_r, x)$. We may now substitute for the coefficients $\dot{p}_r$ and $\dot{r}$. Neglecting again A_r', we obtain $\dot{r} = \frac{p_r}{m}$ from eq. A-2a and $\dot{p}_r = -m\omega_r^2 x$ from eq. 4a. Thus we get for the last two terms of the Boltzman equation

$$\dot{p}_r \frac{\partial \Psi_x'}{\partial p_r} + \dot{r} \frac{\partial \Psi_x'}{\partial x} = -m\omega_r^2 x \frac{\partial \Psi_x'}{\partial p_r} + \frac{p_r}{m} \frac{\partial \Psi_x'}{\partial x} \qquad 2\text{-}8$$

If $\Psi_x' = \Psi_x'(\theta, t, p_\theta, p_r^2 + m^2 w_r^2 x^2)$, eq. 8 equals zero so that the two r terms now give zero in the Boltzman equation which then becomes

$$\frac{\partial \Psi_x'}{\partial t} + \dot{\theta}\,\frac{\partial \Psi_x'}{\partial \theta} + \dot{p}_\theta\left(\frac{\partial \Psi_x'}{\partial p_\theta} + \frac{\partial x}{\partial p_\theta}\,\frac{\partial \Psi_x'}{\partial x}\right) = 0$$

We shall now integrate this equation over $dx\, dp_r$. Thus the first term gives $\frac{\partial \Psi''}{\partial t}$, where $\Psi'' = \int \Psi_x'\, dx\, dp_r$. In the next term we must be more careful as $\dot{\theta} = f(r, p_\theta)$. We will assume that the functional dependence of Ψ_x' on $p_r^2 + m^2\omega_r^2 x^2$ is sharply peaked about $p_r^2 + m^2\omega_r^2 x^2 = 0$. The derivation of the constraint equation shows that if p_r and x are zero initially that they remain very small. Thus Ψ_x' can be a sharply peaked function of these variables. Setting x = 0, now means that $\dot{\theta} = f(p_\theta)$ only because the constraint equation is valid and r is a function of p_θ. Thus integrating the second term we obtain

$$\dot{\theta}\,\frac{\partial \Psi''}{\partial \theta}$$

where $\dot{\theta} = f(p_\theta)$ now. The fourth term is odd in x, because

$$\frac{\partial x}{\partial p_\theta} \cdot \frac{\partial \Psi_x'}{\partial x} = \frac{\partial x}{\partial p_\theta} \cdot \frac{\partial \Psi_x'}{\partial (p_r^2 + m^2\omega_r^2 x)} \cdot \frac{\partial (p_r^2 + m^2\omega_r^2 x)}{\partial x}$$

The first factor is a constant. The second is even in x, while the third is odd. Thus the integral over dx gives zero. This leaves only

$$\dot{p}_\theta\,\frac{\partial \Psi_x'}{\partial p_\theta}$$

Integrating this term now over dx dp_r now gives

$$\dot{p}_\theta\,\frac{\partial \Psi''}{\partial p_\theta}$$

where again we must use the constraint equation to eliminate any r dependence in $\dot{p}_\theta$.

The validity of the reduction of the three-dimensional Boltzman equation to a one-dimensional equation is justified more rigorously in Sec. 2.5, where the complete solution of the Boltzman equation is performed.

If the magnetic field does not satisfy the Betatron 2-1 condition, but still satisfies $0 < n < 1$, then all the results presented in this paper are still correct, because as shown in Appendix II, in such a system

$$\frac{p_\theta}{r} - \frac{e}{c} A_H = \frac{P_s}{r} - \frac{e}{c} A_B$$

where $\mathbf{A}_H$ represents the external field (i. e., mirror fields or synchrotron fields) and $P_s = p_\theta - p_{\theta_0}$. (p_{θ_0} corresponds to an equilibrium orbit at r_0.) Therefore if P_s replaces p_θ in all the equations of this paper, they will still be correct because all the equations of Appendix I are the same.

Sec. 2.2 - The Basic Equations

The Boltzman equation for the system is now one-dimensional. Setting $\Psi'' = \psi$, we have

$$\frac{\partial \psi}{\partial t} + \dot{\theta} \frac{\partial \psi}{\partial \theta} + \dot{p}_\theta \frac{\partial \psi}{\partial p_\theta} = 0 \qquad \text{2-9}$$

where the number of particles in an element $d\theta \, dp_\theta$ is given by

$$dN = \psi \, d\theta \, dp_\theta \qquad \text{2-10}$$

while the coefficients $\dot{\theta}$ and $\dot{p}_\theta$ are defined by eqs. A-2b and A-3b,

$$\dot{\theta} = \frac{1}{r \, m_0 r} \left(\frac{p_\theta}{r} - \frac{e}{c} A_\theta \right) \qquad \text{2-11}$$

$$\dot{p}_\theta = \frac{e\dot{r}}{c}\frac{\partial A_r}{\partial \theta} + \frac{e\dot{z}}{c}\frac{\partial A_z}{\partial \theta} + \frac{e v_\theta}{c}\frac{\partial A_\theta}{\partial \theta} - e\frac{\partial \varphi}{\partial \theta} \qquad \text{2-12}$$

Eqs. 9, 11 and 12 are still too complicated to be solved exactly. We shall, instead, use a perturbation expansion, $\psi = \psi_0 + \psi_1$ where ψ_0 describes the time independent, azimuthally uniform distribution, while ψ_1, contains the θ and t dependence, and is a small quantity compared to ψ_0. This is consistent with our previous approximations. Thus $\frac{\partial \psi_0}{\partial t}$, $\frac{\partial \psi_0}{\partial \theta} = 0$, while $\dot{p}_\theta \sim \psi_1$, by eq. 12.

Thus to terms of lowest order, the Boltzman equation now becomes

$$\frac{\partial \psi_1}{\partial t} + \dot{\theta}\frac{\partial \psi_1}{\partial \theta} + \dot{p}_\theta \frac{\partial \psi_0}{\partial p_\theta} = 0 \qquad \text{2-13}$$

This is called the linearized equation because all terms are linear in ψ_1. Note that $\dot{p}_\theta \frac{\partial \psi_1}{\partial p_\theta}$ has been dropped as it is of second order in ψ_1. The two first terms in eq. 12 may also be dropped as they are of higher order than the last two. More particularly, $A_z = 0$ because the motion in the z direction is symmetric so that $I_z = 0$. The first term may be neglected because by the constraint equation, eq. 6, $\dot{r} \sim \dot{p}_\theta$, while $A_r \sim \psi_1$, so that this term is second order in ψ_1. The coefficients $\dot{\theta}$ and $\dot{p}_\theta$ may now be written more explicitly. They are

$$\dot{\theta} = \frac{1}{\gamma m_0 r}\left(\frac{p_\theta}{r} - \frac{e}{c}\left[A_B + A_\theta'\right]\right) \qquad \text{2-14}$$

where

$$\gamma^2 = \frac{1}{1-\left(\frac{r\dot{\theta}}{c}\right)^2} = \frac{1}{m_0^2 c^2}\left(\frac{p_\theta}{r} - \frac{e}{c}\left[A_B + A'_\theta\right]\right)^2 + 1$$

$$\text{and} \qquad \dot{p}_\theta = \frac{e v_\theta}{c}\frac{\partial A'_\theta}{\partial \theta} - e\frac{\partial \varphi'}{\partial \theta} \qquad \text{2-15}$$

Since $\dot{\theta}$ multiplies ψ_1, in eq. 13, A'_θ gives a second order term and is neglected. The rest of $\dot{\theta}$ is a given function of p_θ and r, and through the constraint equation, $r = f(p_\theta)$, is a function of p_θ only. (A_B is defined by eq. A-4 where now z = 0.) Since p_θ is small, $\dot{\theta}$ may be expanded as a linear function of p_θ as detailed in Appendix IV. The result is

$$\dot{\theta} = \dot{\theta}_0 - k p_\theta \quad , \qquad k = \frac{1}{\gamma m_0 r^2}\left(\frac{1}{1-n} - \frac{1}{\gamma^2}\right) \qquad \text{2-16}$$

This gives one coefficient of the linearized Boltzman equation, eq. 13. The other coefficient is $\dot{p}_\theta$. We desire its explicit dependence on p_θ and ψ_1 also. It now proves more convenient to use Fourier and Laplace transforms as defined by the expressions

$$\begin{Bmatrix}\psi_1^{l\Omega}\\ \dot{p}_\theta^{l\Omega}\end{Bmatrix} = \int_0^\infty dt \int_0^{2\pi} d\theta \; e^{-il\theta + i\Omega t}\begin{Bmatrix}\psi_1(\theta,t)\\ \dot{p}_\theta(\theta,t)\end{Bmatrix} \qquad \text{2-17}$$

which imply the reciprocal relations

$$\begin{Bmatrix}\psi_1(\theta,t)\\ \dot{p}_\theta(\theta,t)\end{Bmatrix} = \frac{1}{(2\pi)^2}\int_L d\Omega \sum_{l=-\infty}^{\infty} e^{il\theta - i\Omega t}\begin{Bmatrix}\psi_1^{l\Omega}\\ \dot{p}_\theta^{l\Omega}\end{Bmatrix}, \qquad \text{2-18}$$

where the contour W is chosen in the upper half-plane parallel to the real Ω -axis above any poles in $\psi_1^{\,\Omega}$, or $\dot{p}_\theta^{\,\Omega}$. See Fig. 3 .
To find $\dot{p}_\theta$ we consider first φ , whose transform is given as a function of ψ by eq. C-20

$$\varphi^{\Omega} = e g_2 \int \psi^{\Omega}(p_0)\, dp_0 \tag{2-19}$$

and is valid only in the beam, and when the wave length of the azimuthal perturbation is much larger than the beam width. Since $\psi = \psi_0 + \psi_1$ and ψ_0 gives rise to an azimuthally symmetric φ , $\frac{\partial \psi}{\partial \theta} = \frac{\partial \psi_1}{\partial \theta}$ and by eq. 19

$$e\left(\frac{\partial \varphi}{\partial \theta}\right)^{\Omega} = \frac{e^2 g_2}{r_0} \int \left(\frac{\partial \psi_1}{\partial \theta}\right)^{\Omega} dp_0 \tag{2-20}$$

Next we examine the expression for A_θ . From eq. A-5

$$A_\theta = \int \frac{v_\theta}{c} \frac{\cos(\theta - \theta')}{|r - r'|} \rho(r, \theta)\, dr\, r\, d\theta$$

$$= e \int \frac{v_\theta}{c} \frac{\cos(\theta - \theta')}{|r - r'|} \psi(\theta, p_0)\, d\theta\, dp_0 \tag{2-21}$$

because the component of $\vec{I}$ along $\vec{A}_\theta$ is I cos ($\theta - \theta'$). This expression for A_θ will be substituted into eq. 15 for $\dot{p}_\theta$ and hence must be evaluated to first order in ψ_1 . Higher order terms will be dropped.
We need first an explicit expression for $v_\theta = f(p_0)$, valid to first order in ψ_1 , where $v_\theta = r\dot{\theta}$. For r, the constraint eq. 6 gives,

$$r - r_0 = \alpha_p p_0 \tag{2-22}$$

In the expansion of $\dot{\theta}$ in Appendix IV, some changes must be made as we desire $\dot{\theta}$ to first order. We now include A'_θ, so that the expansion will have $\frac{p_\theta}{r} - \frac{e}{c}A'_\theta$ as a factor. Also we will expand the r dependence separately to lowest order in $r - r_0$ and then substitute eq. 22. Thus using eq. 14 instead of eq. D-1 and D-3 in Appendix IV we obtain

$$\dot{\theta} = \dot{\theta}_0 + \frac{1}{m_0 r_0^2 \gamma^3}\left(p_\theta - r_0 \frac{e}{c} A'_\theta\right) - \dot{\theta}_0 (\alpha_p p_\theta)$$

and to first order terms,

$$v_\theta = r\dot{\theta} = (r_0 + \alpha_p p_\theta)\left[\dot{\theta}_0 + \frac{1}{m_0 r^2 \gamma^3}\left(p_\theta - r_0 \frac{e}{c} A'_\theta\right) - \frac{\dot{\theta}_0}{r_0}(\alpha_p p_\theta)\right]$$

$$\approx r_0 \dot{\theta}_0 + \frac{1}{m_0 r_0^2 \gamma^3}\left(p_\theta - r_0 \frac{e}{c} A'_\theta\right) \qquad \text{2-23}$$

$$= v_0 + \frac{1}{\gamma^3 m_0}\left(\frac{p_\theta}{r_0} - \frac{e}{c} A'\right)$$

This term together with $\psi = \psi_0 + \psi_1$, must be inserted in eq. 21. Keeping only terms first order in ψ_1, we obtain from eqs. 21 and 23

$$A'_\theta = \int\left[\left(\frac{v_0}{c} + \frac{p_0}{m_0 c \gamma^3 r_0}\right)\psi_1 - \frac{e A' \psi_0}{m_0 c \gamma^3}\right] \cos(\theta - \theta') \frac{d\theta \, dp_\theta}{|r - r'|} \qquad \text{2-24}$$

The θ dependence of the bracket is only in ψ_1 and A', as $A' \sim \psi_1$. The results of Appendix III, in particular eq. C-20, now gives

$$A_\theta^{\prime \ell n} = \frac{e \hat{g}_\ell}{r_0} \int\left[\left(\frac{v_0}{c} + \frac{p_\theta}{m_0 c \gamma^3 r_0}\right)\psi_1^{\ell n} - \frac{e A'^{\ell n} \psi_0}{m_0 c \gamma^3}\right] dp_\theta \qquad \text{2-25}$$

The ψ_0 integration may be easily performed. Since ψ_0 is normalized to N on the field θ and p_θ and is constant from $\theta = 0$ to 2π, $\int \psi_0 \, dp_\theta = \frac{N}{2\pi}$ where N is the total number of particles. Thus transposing and dividing

$$\left(\frac{\partial A_0'}{\partial \theta}\right)^{in} = \frac{1}{1 + \frac{\nu \hat{g}}{\gamma^3}} \frac{e \hat{g}}{r_0} \int \left(\frac{v_0}{c} + \frac{p_0}{c m_0 \gamma^3 r_0}\right)\left(\frac{\partial \psi_1}{\partial \theta}\right)^{in} dp_0 \qquad \text{2-26}$$

Substituting 20, 23 and 26 into eq. 15 now gives,

$$\dot{p}_0^{\,in} = e\left(\frac{v_0}{c} + \frac{p_0}{c m_0 \gamma^3 r_0}\right) \frac{e \hat{g}}{r_0} \frac{1}{1 + \frac{\nu \hat{g}}{\gamma^3}} \int \left(\frac{v_0}{c} + \frac{p_0}{c m_0 \gamma^3 r_0}\right)\left(\frac{\partial \psi_1}{\partial \theta}\right)^{in} dp_0$$

$$- \frac{e^2 g}{r_0} \int \left(\frac{\partial \psi_1}{\partial \theta}\right)^{in} dp_0 \qquad \text{2-27}$$

As we will show, the p_θ terms in eq. 25 may be neglected. Eq. 27 is then proportional to

$$1 - \left(\frac{v_0}{c}\right)^2 \frac{\frac{\hat{g}}{g}}{1 + \frac{\nu \hat{g}}{\gamma^3}} \approx \frac{1}{\gamma^2}\left(1 + \frac{\nu \hat{g}}{\gamma}\right) - \frac{v_0^2}{c^2}\frac{\delta g}{g} \approx \frac{1}{\gamma^2} - \frac{v_0^2}{c^2}\frac{\delta g}{g} \equiv \frac{1}{\gamma_g^2} \qquad (3)$$

(where $\delta g = \hat{g} - g > 0$ and $\frac{\nu g}{\gamma} \ll 1$ from the Schmidt criterion for the neglect of the zero-order self-field.) The term $\frac{p_0}{c m_0 \gamma^3 r}$ outside the integral is negligible if $p_0 \ll \gamma m_- c r$ or $\Delta v_- \ll \frac{c}{\gamma^2}$, while for the positive specie $\Delta v_+ \ll c$ is sufficient, if the ions are non-relativistic. Now consider the p_θ term under the integral. By eq. 31, ψ_1 consists of two parts. If the part proportional to R (ℓ), (eqs. 30 and 31), is sufficiently smooth and involves only small p_θ, so that $p_0 \ll \gamma m_- c r_0$, ($p_0 \ll m_+ c r_0$ for the ions) this part may be neglected. Using the other part of ψ_1, and the identity of eq. 3-32, p_θ may be taken out of the integral and equals $\frac{\dot\theta - \frac{\Omega}{l}}{k}$. Thus substituting for 'k' from eq. 16, if $\left|\dot\theta_- - \frac{\Omega}{l}\right| \ll \frac{1}{1-h}\frac{c}{r_0}$ this term may be neglected. For the positive specie, the criteria $\left|\dot\theta_+ - \frac{\Omega}{l}\right| \ll \frac{h}{1-h}\frac{c}{r_0}$ is obtained, which is well satisfied, because the constraint equation must be valid and hence eq. 7 must be satisfied.

Thus eq. 27 becomes

$$\dot p_0^{\,l\Omega} = -\frac{e^2 g_1}{r_g^2 r_0}\int\left(\frac{\partial \psi_1}{\partial \theta}\right)^{l\Omega} d\,p_0\,, \qquad 2\text{-}28$$

$$\frac{1}{r_g^2} \equiv \frac{1}{r^2} - \frac{v^2}{c^2}\left(\frac{\hat g - g}{g}\right)$$

This equation together with eqs. 13 and 16 are completely equivalent to the plasma system first solved correctly by L. D. Landau.(18)

Sec. 2.3 - The Dispersion Relation

We solve the equations using Fourier and Laplace transforms following J. D. Jackson's(21) notation and method. We multiply eq. 13 by $\int_0^{2\pi} d\theta \int_0^{\infty} dt\, e^{-i(l\theta - \Omega t)}$ to obtain after integrating by parts,

$$\int_0^{2\pi} d\theta \left[\psi_1(\theta,t)\, e^{-il\theta + i\Omega t}\right]_0^{\infty} - i(\Omega - l\dot\theta)\int \psi_1(\theta,t)\, e^{-il\theta + i\Omega t}\, d\theta\, dt \qquad 2\text{-}29$$

$$+ \frac{\partial \psi_0}{\partial p_0}\int \dot p_0\, e^{-il\theta + i\Omega t}\, d\theta\, dt = 0$$

On the assumption that Ω has a positive imaginary part, the first term evaluated at $t = \infty$ vanishes and the rest is $-R(l)$ where

$$R(l) = \int_0^{2\pi} e^{-il\theta}\, \psi_1(\theta, t=0)\, d\theta \qquad 2\text{-}30$$

i.e., the Fourier transform of the initial displacement. With the definitions of the Fourier transforms as given by eqs. 17 and 18, eq. 29 now gives the result

$$\psi_1^{\eta\Omega} = \left(R(\eta) - \frac{\partial \psi_0}{\partial p_0} \dot{p}_\theta^{\eta\Omega} \right) \frac{1}{i(\eta\dot{\theta} - \Omega)} \qquad \text{2-31}$$

From eq. 17, by integrating by parts, $\left(\frac{\partial \psi_1}{\partial \theta}\right)^{\eta\Omega} = i\eta\, \psi^{\eta\Omega}$ and thus eq. 28 becomes

$$\dot{p}_0^{\eta\Omega} = - \frac{e^2 g}{r_0 \gamma_g^2} (i\eta) \int \psi_1^{\eta\Omega}\, dp_0 \qquad \text{2-32}$$

Inserting eq. 31 into eq. 32, now gives

$$\dot{p}_\theta^{\eta\Omega} = \frac{e^2 g}{r_0 \gamma_g^2} \int \frac{\frac{\partial \psi_0}{\partial p_0} \dot{p}_0^{\eta\Omega} - R(\eta)}{\dot{\theta} - \frac{\Omega}{\eta}}\, dp_0 \qquad \text{2-33}$$

Since $\dot{p}_\theta^{\eta\Omega}$ is independent of p_θ, we can solve eq. 33 for $\dot{p}_\theta^{\eta\Omega}$,

$$\dot{p}_0^{\eta\Omega} = \frac{\frac{e^2 g}{r_0 \gamma_g^2} \int \frac{R(\eta)}{\dot{\theta} - \frac{\Omega}{\eta}}\, dp_0}{1 - \frac{e^2 g}{r_0 \gamma_g^2} \int \frac{\frac{\partial \psi_0}{\partial p_0}}{\dot{\theta} - \frac{\Omega}{\eta}}\, dp_0} \equiv \frac{\Phi(\eta, \Omega)}{H\left(\frac{\Omega}{\eta}\right)} \qquad \text{2-34}$$

where Φ and H are the numerator and denominator of eq. 34 respectively. The inverse $\dot{p}_0^{\eta}$, which gives the time behavior of $\dot{p}_0^{\eta}$ is obtained from eq. 30 and is, using eq. 34

$$\dot{p}_o^{\,l} \equiv \int_W e^{-i\Omega t}\, d\Omega\, \dot{p}_o^{\,l\Omega}$$

$$= \int_W e^{-i\Omega t}\, d\Omega\, \frac{\Phi(l,\Omega)}{H\left(\frac{\Omega}{l}\right)} \tag{2-35}$$

Since the curve W may be closed in the lower half-plane of Ω (see Fig. 3), we may evaluate the integral using residues, and obtain

$$\dot{p}_o^{\,l} = 2\pi i \sum_m e^{-i\Omega_m t}\, \mathrm{Res}\left[\frac{\Phi(l,\Omega)}{H\left(\frac{\Omega}{l}\right)}\right]_{\Omega=\Omega_m} \tag{2-36}$$

where the sum is over the poles of the term in brackets. If the initial perturbation is sufficiently smooth, then $\Phi(l,\Omega)$, will not contribute any poles and the poles will occur only for the zeros of the denominator. If the poles have a positive imaginary part, then $\dot{p}_o^{\,l} \sim e^{\alpha t}$ and the solution is unstable. Since the above function $H\left(\frac{\Omega}{l}\right)$ is defined only for Ω with positive imaginary part as appears from eq. 29, we must find the analytic continuation of $H\left(\frac{\Omega}{l}\right)$ in the integrand of eq. 35 in order to find the residues. To find $H\left(\frac{\Omega}{l}\right)$ explicitly we must insert the value of $\dot{\theta}$ from eq. 16. We then see that $H\left(\frac{\Omega}{l}\right)$ is not continuous across the real axis (viewed as a function in the complex Ω plane), because

$$\lim_{\frac{\Omega}{l} \to u \pm i\epsilon} \frac{1}{(\dot{\theta}_o - k p_o) - \frac{\Omega}{l}} = P\frac{1}{(\dot{\theta}_o - k p_o) - u} \pm \pi i\, \delta\left[(\dot{\theta}_o - k p_o) - u\right] \tag{2-37}$$

The δ function gives the discontinuity. This shows that H, defined by eq. 34, is discontinuous across the real axis with a jump equal to

$$\Delta H = H\left(\frac{\Omega}{l} = u + i\epsilon\right) - H\left(\frac{\Omega}{l} = u - i\epsilon\right)$$

$$= -2\pi i \frac{e^2 g}{k r_0 \gamma_g^2} \frac{\partial \psi_0}{\partial p_0}\left(\frac{-\frac{\Omega}{l} + \dot{\theta}_0}{k}\right) \leftarrow \text{arg of } \psi_0 \qquad \text{2-38}$$

where we have used the δ function properties $\delta(x) = \delta(-x)$, $\delta(ax) = \frac{\delta(x)}{a}$ and recalled that the integration variable in H is p_θ. Since the analytic continuation of H must be continuous, we add ΔH to H defined by eq. 34 to get the form of this function valid in the lower half-plane. The dispersion relations are now given by setting H = 0. Thus

$$H\left(\frac{\Omega}{l}\right) = 0 = 1 - \frac{e^2 g}{r_0 \gamma_g^2} \int \frac{\frac{\partial \psi_0}{\partial p_0}}{\dot{\theta} - \frac{\Omega}{l} - k p_0} \, dp_0 \qquad \operatorname{Im} \Omega > 0$$

$$\text{2-39}$$

$$0 = 1 - \frac{e^2 g}{r_0 \gamma_g^2} \int \frac{\frac{\partial \psi_0}{\partial p_0} \, dp_0}{\dot{\theta}_0 - \frac{\Omega}{l} - k p_0} - 2\pi i \frac{e^2 g}{k r_0 \gamma_g^2} \frac{\partial \psi_0}{\partial p_0}\left(\frac{-\frac{\Omega}{l} + \dot{\theta}_0}{k}\right) \qquad \operatorname{Im} \Omega < 0$$

In eq. 39 it has been assumed that $l > 0$. If $l < 0$ then it may be seen that the first equation is unchanged, but that in the second there is a plus sign in front of the $2\pi i$ in the third term on the right hand side.

Sec. 2.4 - Stability Criteria and Dispersion Relation Solutions

In this section we will summarize some results obtained by investigating the solution of the dispersion relations. If one desires to know only stability criteria, i.e., the condition that Ω in eq. 18 has a negative imaginary part, so that the corresponding Fourier component is exponentially damped, then it is unnecessary to solve the dispersion relations completely. By means of the Nyquist diagram as elaborated by Penrose[22] and Jackson,[21] it is possible to answer the stability question by only evaluating certain integrals. As shown in Appendix V, the number of integrals equals the number of maximum and minimum of the distribution function. For a zero-order distribution function with one maximum, the stability criterion, from Appendix V is

$$(\Delta p)^2 > \left(2\sqrt{\frac{\nu g \gamma}{\gamma_g^2 \alpha}}\; m_0 c\, r_0\right)^2 \qquad 2\text{-}40$$

where

$$\frac{4}{(\Delta p)^2} = \frac{2\pi}{N}\int \frac{\psi_0^{\circ} - \psi_0}{p_{0_0} - p_0}\, dp_0 \qquad 2\text{-}41$$

and ψ_0°, p_{θ_0} are the values at the maximum. Since $\left.\frac{\partial \psi_0}{\partial p}\right)_{p_{0_0}} = 0$ the integral is not singular.

For a Maxwellian distribution

$$\psi_0 = \frac{N}{2\pi}\left(\frac{1}{\Delta_m\sqrt{\pi}}\, e^{-\frac{p^2}{\Delta_m^2}}\right) \qquad 2\text{-}42$$

we find from Appendix V, that $\Delta p = \sqrt{2}\,\Delta_m$. This together with eq. 40 gives the stability criterion.

For a resonance distribution

$$\psi_0 = \frac{N}{2\pi}\left(\frac{\Delta_r}{\pi} \quad \frac{1}{p^2 + \Delta_r^2}\right) \qquad 2\text{-}43$$

it is possible to solve the set of equations 39 exactly. The details are in Appendix VI. The result is

$$\frac{\Omega}{l} - \dot{\theta}_0 = \frac{-i\alpha}{\gamma m_0 r_0^2}\left(\Delta_r \pm \sqrt{\frac{\nu g \gamma}{\gamma_g^2 \alpha}}\; m_0 c r_0\right) \qquad 2\text{-}44$$

When l is negative then $i \rightarrow -i$. The criterion for stability is similar to that of the Maxwellian distribution. The physical meaning of the stability criterion is the following. For a cold beam there is instability if the particles have a negative effective mass. This means if $\gamma_g^2 \alpha > 0$ Suppose then that this is true. Then the growth rate of the instability is given by the second term on the R.H.S. of eq. 44. The first term on the R.H.S. gives the spread in angular velocity of the components of the beam due to the spread in p_θ. The stability criterion means now that if the spread in angular velocity is greater than the growth rate, there will be stability, because the particles will have mixed themselves during the characteristic growth time so that any perturbation will have been washed out. This effect, where a finite temperature effects stability, occurs in many plasma physics problems including for example the two-stream instability.

Eq. 44 also shows that when there is stability, the oscillations are damped. This effect has not been previously noticed. The result, eq. 44, without the damped solution is similar to the result of Kolomenskii and Lebedev.[9] This damping is due to the additional term in the second of eq. 2-39, which in the case of one-dimensional plasmas gives Landau damping.[18] For a pulse function this term is zero, because $\frac{\partial \psi_0}{\partial p} = 0$, and therefore N-S[8] who used pulse functions did not find this damping. We note that also in the case of one-dimensional plasmas, the use of pulse functions leads to no damping and, in fact gives the fluid equations for longitudinal oscillations, which are known not to exhibit damping.

Finally we use the results of Appendix VI, which gives the solution of the dispersion relation for a pulse function of width Δ ,

$$\frac{\Omega}{l} - \dot{\theta}_0 = \pm \frac{\alpha}{\gamma m_0 r_0^2} \sqrt{\frac{\Delta^2}{4} - \frac{\nu g \gamma}{\gamma_g^2 \alpha} (m_0 c r_0)^2} \qquad 2\text{-}45$$

This result is similar to that obtained by N-S[8] and Kolomenskii and Lebedev.[9] If $\gamma_g^2 = \gamma^2$, then the result is the same. This occurs only when the beam is sufficiently thin, for then g gets large and $\frac{\delta g}{g} \to 0$. It is also true for large wave numbers for then $\delta g \to 0$, as is apparent from Appendix III. Since, by its definition in eq. 28, γ_g^2 may be negative for low wave numbers, we see by eqs. 40, 44 and 45 that even for a cold beam there is no N.M.I. for weakly relativistic beams. As an example take the case $l = 3$, $\delta g, \approx .1$, $g \approx 5$

(δq_l is obtained from eq. C-13) where

$$\frac{1}{r_g^2} = \frac{1}{j^2} - \frac{1}{5}$$

For $r > 7$, the three lowest wave numbers are stable. For $l = 1$, $\delta q_2 = 1.3$. In general, δq_l is a function of l only. Our result differs from the other authors because the beam is free and is not in a vacuum tank which would shield parts of the beam from another. Another contribution is that the validity of the constraint equation, requires, by eq. 7, that the R.H.S. of eq. 45 be much less than $\frac{\sqrt{1-n}}{l}\dot{\theta}_0$. Since the beam minor radius is much less than the major radius, eq. 6, the constraint equation, implies that the first term in eq. 45 is much less than $\dot{\theta}_0$. Hence the second term must also be less than $\dot{\theta}_0$.

If $\Delta = 0$, the instability growth rate is

$$\frac{\Omega}{l} - \dot{\theta}_0 = \pm i \sqrt{\frac{\nu q \alpha}{r_g^2 r}} \frac{c}{r} \qquad \text{2-46}$$

and is valid only if this quantity is much less than $\dot{\theta}_0$.

From the above results, it is apparent that the criteria for stability are insensitive to the precise shape of the distribution. Thus the pulse function distribution, which is simplest for computations, gives adequately accurate results. We suspect that this holds true for many calculations in plasma physics, where the utility of the pulse function is insufficiently appreciate.

In systems of azimuthal symmetry, p_θ is a constant of the motion. Thus neglecting any azimuthal instabilities, in a device, for example, like the betatron, Δp_θ in eq. 40, may be calculated from

Δp_θ of the particles at the time of injection, as it is constant. When the external field is zero $p_\theta = m_0 v_\theta r$. Since $r \approx r_0$, the spread in p_θ comes mostly from the spread of v_θ. The number of particles in a range dv_θ is $N_v = \frac{m_0}{(2\pi kT)^{1/2}} e^{-\frac{m_0 v_\theta^2}{2kT}}$. Hence the number of particles in a range $d\,p_\theta$ is

$$N_p = \left(\frac{1}{\pi \cdot 2kT m_0 r_0^2}\right)^{1/2} e^{-\frac{p_\theta^2}{2m_0 kT r_0^2}}$$

and comparing with eq. 42, $\Delta_m = \sqrt{2m_0 kT r_0^2}$. Inserting into eq. 40, we obtain

$$\frac{\nu\, r\, g}{r_g^2\, \alpha} < (2.0 \times 10^{-6}) \times T_{e.v.} \qquad \text{2-47}$$

where the electron temperature is expressed in electron volts. This expression is also valid for the ions if ν is replaced by $\nu_+' = \frac{N_+}{2\pi r_0} \frac{e^2}{m_+ c^2}$. Thus for a plasma at a given temperature, that is non-relativistic, the number of ions or electrons that are stable, is the same. The quantity α is $\alpha = \frac{1}{1-n} - \frac{1}{\gamma^2}$ and 'g' is given in Appendix III. For ions or non-relativistic electrons, some typical parameters are $\alpha \approx 1$, $g \approx 5$, $T \approx 3\,e.v.$. Eq. 47 then gives $\nu < 1.2 \times 10^{-6}$ which corresponds to 1/50 amp of relativistic electrons.

Sec. 2.5 - The Effect of Betatron Oscillations

a. Axial (z) Oscillations

We shall next investigate the effect of allowing small amplitude betatron oscillations in the zero-order distribution function. To simplify the investigation we shall consider first the (z) oscillations and then the (r) oscillations. We will show that these oscillations have a small effect and give a negligible contribution to stability.

This result differs from the investigations at MURA as reported by Nielsen et al.[30] They define two quantities ΔE_p and ΔE_z, which give the spread in beam energy due to a spread in p_θ (ΔE_p) and a spread in betatron oscillation amplitude (ΔE_z). These quantities may be obtained by expansion of the Hamiltonian, eq. A-1, since these energy spreads are small. The quantity $\Delta E_p \sim \Delta$ and $\Delta E_z \sim \frac{\rho^2}{r_0^2}$ Therefore the square root of the L.H.S. of eq. 40 is proportional to ΔE_p. ($\gamma_g^2 = \gamma^2$ in eq. 40 to correspond with Nielsen's equations.) Nielsen et al. now remark that the effect of the betatron oscillations is to add a ΔE_z term to the ΔE_p term, so that both terms contribute equally to stability . This result is clearly different from our result, given below, eq. 57.

By neglecting the p_r and r terms in the Boltzman equation, eq. 1, following the constraint equation arguments given in Sec. 2.1, that equation becomes after linearization,

$$\frac{\partial \psi_1}{\partial t} + \dot{\theta}\frac{\partial \psi_1}{\partial \theta} + \dot{p}_\theta \frac{\partial \psi_0}{\partial p_\theta} + \dot{p}_z \frac{\partial \psi_1}{\partial p_z} + \dot{z}\frac{\partial \psi_1}{\partial z} = 0 \qquad \text{2-48}$$

This differs in a few respects from eq. 13. First, there are two new terms in $\dot{p}_z$ and $\dot{z}$. Next ψ_1 now contains a factor describing particles of small, but finite amplitude oscillations. In addition $\dot{\theta}$ is not given by eq. 16, but now contains an additional term proportional to z^2, which results from retaining the z^2 term from A_θ in eq. D-1. Thus:

$$\dot{\theta} = \dot{\theta}_0 - k p_\theta + \dot{\theta}_0 \frac{n z^2}{2 r_0^2} \tag{2-49}$$

We shall therefore be examining whether this additional variation in $\dot{\theta}$ will add to the variation in $\dot{\theta}$ due to p_θ and hence add an additional term for stability in eq. 39. We also need the value of the other coefficients in eq. 48. We have from eq. A-3b, for the non-relativistic case

$$\dot{p}_\theta = - e \frac{\partial \varphi}{\partial \theta} \quad . \tag{2-50}$$

Keeping only first order terms in the coordinates we obtain $\dot{p}_z$ from eq. A-3c,

$$\dot{p}_z = \frac{e v_0}{c} \left(- \frac{m c v_0}{e} \right) \frac{n z}{r_0^2} = - \frac{m v_0^2 n}{r_0^2} z \quad . \tag{2-51}$$

From eq. A-2a

$$\dot{z} = \frac{p_z}{m} \quad . \tag{2-52}$$

We shall also assume that

$$\left. \begin{matrix} \psi_1 \\ \varphi \end{matrix} \right\} \sim \left\{ \begin{matrix} \psi_{1n} \\ \varphi_{1n} \end{matrix} \right. e^{i(l\theta - \Omega t)} \tag{2-53}$$

This will give the same stability criteria as an initial value problem. Eq. 48 now becomes

$$(a_0 + b z^2) \psi_{1n} - a_1 \varphi_{1n} \frac{\partial \psi_0}{\partial p_\theta} - a_2 z \frac{\partial \psi_{1n}}{\partial p_z} + a_3 p_z \frac{\partial \psi_{1n}}{\partial z} = 0 \tag{2-54}$$

where the a's and b are constants independent of p_z, z, and are

$$a_0 = -i(\Omega - l\dot{\theta}_0 + lk p_0)$$

$$b = il\dot{\theta}_0 \frac{h}{2r_0^2} \qquad a_2 = \frac{m v_0^2 h}{r_0^2} \qquad \text{2-55}$$

$$a_1 = eil \qquad a_3 = \frac{1}{m}$$

We note that we are forming two separate expansions. One is of ψ in powers of φ. The other is in powers of z, p_θ, p_z. Thus since we have kept only the first order terms in z and p_z for the last two terms in eq. 54 to be consistent we must do the same in the equation for ψ_0. Thus since the zero-order Boltzman equation is

$$\dot{p}_z \frac{\partial \psi_0}{\partial p_z} + \dot{z} \frac{\partial \psi_0}{\partial z} = 0 \qquad \text{2-56}$$

we have

$$-a_2 z \frac{\partial \psi_0}{\partial p_z} + a_3 p_z \frac{\partial \psi_0}{\partial z} = 0$$

Note that ψ_0 is assumed independent of θ and t and therefore the other terms of eq. 56 are zero. The solution of eq. 56 is evidently:

$$\psi_0 = A f(a_2 z^2 + a_3 p_z^2) \qquad \text{2-57}$$

where A may be a function of p_ϕ.

We have now given eq. 44 with the z and p_z dependence of all the terms given explicitly. We also know that the solution of eq. 54 when the terms approach zero is given by eq. 31 with R (l) = 0. This is sufficient to solve the equation.

We have ignored self-field terms in the expressions for $\dot{p}_z$ and $\dot{z}$ that give first order terms in the perturbation amplitude in eq. 48. $\dot{z}$ has the self-field term A'_z whose neglect may be justified because the use of ψ_1 obtained from the solution of eq. 54 gives zero for A'_z. This justifies the remark made above, that I_z remains zero during the N.M.I. The dominant self-field term in $\dot{p}_z$ is $\frac{\partial \varphi'}{\partial z}$, which has the same z dependence as the dominant external vector potential, and is therefore ignored. This means that we neglect the transverse space charge force.

The solution of eq. 54 proceeds in a straightforward way using the method of characteristics. This solution is then inserted into eq. C-2 to obtain the dispersion relation. These details are in Appendix VIII. The solution obtained there for ψ_{1n} has been verified by insertion into eq. 54 directly. The result is that ψ_0 in eq. 39a is multiplied by a factor

$$1 - .005\,\eta_n^2\,\frac{\rho^4}{r_0^4}$$

This is equivalent to multiplying ν by this factor. Examination of eqs. 40, 44 or 45 shows that this has a negligible effect on stability. Thus the inclusion of the axial betatron oscillations does improve stability, but negligibly.

b. <u>The Radial (r) Oscillations</u>

Using the above procedure for evaluating the z betatron oscillations we shall now evaluate the effect of the radial betatron oscillations on the N.M.I. We shall, for simplicity ignore the z oscillations in our treatment.

The linearized Boltzman equation is then:

$$\frac{\partial \psi_1}{\partial t} + \dot{\theta}\frac{\partial \psi_1}{\partial \theta} + \dot{p}_\theta \frac{\partial \psi_1}{\partial p_\theta} + \dot{r}\frac{\partial \psi_1}{\partial r} + \dot{p}_r \frac{\partial \psi_1}{\partial p_r} = 0 \qquad \text{2-58}$$

This is just like eq. 48, but now we have substituted r for z. We now write the coefficients. $\dot{\theta}$ now has first order terms in $r - r_0$, so we shall drop the second order terms. By eq. A-2b

$$\dot{\theta} = \frac{1}{mr}\left(\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right) .$$

Expanding and keeping only first order terms we obtain

$$\dot{\theta} = \dot{\theta}_0 + \frac{p_\theta}{m r_0^2} - \frac{\dot{\theta}_0 (r - r_0)}{r_0} \qquad \text{2-59}$$

$\dot{p}_\theta$ is given by eq. 50 and $\dot{r}$, like eq. 52 is obtained from eq. A-2a, giving $\dot{r} = \frac{p_r}{m}$. Again keeping only first order terms, we obtain $\dot{p}_r$ from eq. A-3a

$$\dot{p}_r = \frac{v_0}{r_0^2} p_\theta - \frac{m v_0^2}{r_0^2}(1-n)(r - r_0) \qquad \text{2-60}$$

To simplify the calculations, we will define

$$r - r_e \equiv r - r_0 - \frac{p_\theta}{m v_0 (1-n)} \qquad \text{2-61}$$

Using eq. 61 to simplify eqs. 59 and 61, and using also the θ, t dependence of eq. 53, the linearized Boltzman equation takes the form

$$(a_0 + bx)\psi_{1n} - a_1 \varphi_{1n}\frac{\partial \psi_0}{\partial p_\theta} - a_2 x \frac{\partial \psi_{1n}}{\partial p_r} + a_3 p_r \frac{\partial \psi_{1n}}{\partial x} = 0 \qquad \text{2-62}$$

where the a's and b are constants independent of r, p_r and are

$$a_0 = -i(\Omega - l\dot{\theta}_0 + lk p_\theta) \qquad a_2 = m\frac{v_0^2}{r_0^2}(1-n)$$

$$b = -il\frac{\dot{\theta}_0}{r_0} \qquad a_3 = \frac{1}{m} \tag{2-63}$$

$$a_1 = eil \qquad x = r - r_e$$

Since eq. 62 is a partial differential equation in r, p_r, we may consider p_θ = constant in eq. 62 and hence dr = dx. k is defined in eq. D-6. We note again that we are forming two expansions. One is of ψ in powers of φ, and the other is of $\dot{p}_i$, $\dot{r}_i$ in powers of $r - r_0$, p_θ, p_r. Thus since we have kept only first order terms in x, p_r for the last two terms of eq. 62, to be consistent we must do the same in the Boltzman equation for ψ_0. Thus

$$\dot{p}_r \frac{\partial \psi_0}{\partial p_r} + \dot{r}\frac{\partial \psi_0}{\partial r} = 0$$

or

$$-a_2 x \frac{\partial \psi_0}{\partial p_r} + a_3 p_r \frac{\partial \psi_0}{\partial x} = 0 \tag{2-64}$$

and

$\psi_0 = A_p f(a_2 x^2 + a_3 p_r^2)$ is a solution of eq. 64, 2-65

where A_p may only be a function of p_θ. Knowing the form of ψ_0 given in eq. 65, and the dispersion relation obtained when the $r - r_0$, p_r terms approach zero (as given by eq. 39), we can now proceed to solve eq. 62.

In writing eq. 62, the first-order, self-field terms A_r', A_θ', A_z', φ' in $\dot{r}$ and $\dot{p}_r$ which contribute to the first order Boltzman equation, have not been included. This may be justified in an approximate way. Consider first $\dot{r}$. By eq. A-2a, this contains a term A'_r which is found to lowest order by integrating ψ_1, with a factor p_r. Looking at the solution in Appendix IX found by neglecting A'_r, we find that A'_r is indeed small, for the contribution to A'_r is obtained only from a term odd in p_r and even in x. The largest such term is the third term in eq. I-13 or 14, and therefore A'_r has a factor $\frac{\rho^2}{r^2}$, which allows us to neglect it. This means that the perturbed vector potential is small because the transverse currents that cause it are due to particles which have slow transverse velocity relative to the angular velocity.

The other neglected self-field terms are in $\dot{p}_r$. The largest of these is $\frac{\partial\varphi'}{\partial r}$. The major contribution to this term will be a term proportional to x, which is the same spatial dependence as the external field. Thus since the zero-order potential $\frac{\partial\varphi^o}{\partial r}$, is assumed negligible, we shall also neglect this term. This means that we are neglecting the transverse space charge forces.

The solution of eq. 62 is carried out in Appendix IX by the method of characteristics. This solution has been verified by insertion directly into eq. 62. The result is that ψ_0 in eq. 39a is multiplied by a factor

$$\frac{1 - \frac{\eta^2}{8}\frac{1}{1-n}\frac{\rho^2}{r_0^2}}{1 + \frac{3}{2}\frac{\nu g c^2}{r_0^2}\frac{\eta^2}{(1-n)^2\dot{\theta}_0^2}}$$

As is more clearly seen from the solution I-22, this shows that the effect of the radial betatron oscillations on stability is also quite small.

The denominator is unity if eq. 7 , the condition for the validity of the constraint equation, is true. This formula thus suggests that as the constraint equation is violated, the instability growth rate slows down, presumably because energy is also being put into the excitation of radial betatron oscillations.

CHAPTER III - TWO STREAMS

In this chapter the dispersion relation for the N.M.I. of counterstreaming positive ions and relativistic electrons is derived. The same formalism is used to derive the dispersion relation of the longitudinal oscillations of thin beams, which is different from that derived by Finkelstein and Sturrock.[5] To show the consistency of this formalism, the dispersion relation for infinitely wide streams is also found and agrees with that found by Bludman et al.[19]

Sec. 3.1 - The Negative Mass Instability Equations

In order to find the dispersion relation for the N.M.I. for two streams, it is necessary to write the linearized Boltzman equation for each stream. Using the same approximations as made above, in Sec. 2.1 and 2.2, where the single stream is discussed, we obtain again eqs. 2-13 and 2-16. The restriction on ν may, however, be greatly relaxed if we assume a neutralized beam, for then there is no zero order transverse electric potential. There will, though, be a first order electric potential due to the fact that under the N.M.I. there is a transverse motion of unequal amount for each stream. However this term, of first order in the perturbation amplitude ($\sim \psi_1$) if inserted into the constraint equation through eq. A-3a, will give a second order term in the one-dimensional Boltzman equation 2-13, and hence is neglected.

The procedure is to solve the linearized Boltzman equation for each stream and use this perturbed distribution function to obtain the perturbed potentials, A' and φ '. These are inserted into the equations for $\dot{p}_\theta$ for each beam. This results in two equations in the two unknowns $\dot{p}_\theta$ + and $\dot{p}_\theta$ -. The requirement that there be a solution implies that a certain determinant be zero, which gives the dispersion relation.

In stead of solving the Maxwell-Boltzman equations of Sec. 2.2 as an initial value problem as done in Sec. 2.3, we will assume that all the perturbed quantities vary as $e^{il\theta - i\Omega t}$. This will give us the correct dispersion relation for the growing solutions and will therefore be sufficient to find stability criteria. The Boltzman equation, eq. 2-13 now gives after solving for $\psi_{l\Omega}$,

$$\psi_{\pm}^{l\Omega} = \frac{-\dot{p}_{\pm}^{l\Omega}\,\dfrac{\partial \psi_{0\pm}}{\partial p_\theta}}{il\left(\dot{\theta}_{\pm} - \dfrac{\Omega}{l}\right)} \qquad 3\text{-}1$$

Next $\dot{p}_\theta$ must be evaluated to first order. By eq. 2-15 we have for particles of charge e and velocity v_θ

$$\dot{p}_\theta = e\,\frac{v_\theta}{c}\,\frac{\partial A_\theta}{\partial \theta} - e\,\frac{\partial \varphi}{\partial \theta} \qquad 3\text{-}2$$

By eq. 2-25, we have for A'_θ, neglecting p_θ as justified in Sec. 2.2,

$$\frac{\partial A'_\theta}{\partial \theta} = \frac{e\hat{g}}{r_0}\,\frac{\partial}{\partial \theta}\int\left(\frac{v_+}{c}\,\psi_{1+} - \frac{e}{m_+c^2}\,A'_\theta\,\psi_{0+}\right)dp_\theta \qquad 3\text{-}3$$

$$- \frac{e\hat{g}}{r_0}\,\frac{\partial}{\partial \theta}\int\left(\frac{v_-}{c}\,\psi_{1-} - \frac{e}{\gamma^3 m_- c^2}\,A'_\theta\,\psi_{0-}\right)dp_\theta$$

Since A'_θ is independent of p_θ and ψ_0 is normalized to $\frac{N}{2\pi}$, we obtain

$$\frac{\partial A'_\theta}{\partial \theta} = \frac{1}{1 + \nu \hat{g} \frac{m_-}{m_+} + \frac{\nu \hat{g}}{\gamma^2}} \frac{e \hat{g}}{r_0} \frac{\partial}{\partial \theta} \int \left(\frac{v_+}{c} \psi_{1+} - \frac{v_-}{c} \psi_{1-} \right) d p_\theta \qquad 3\text{-}4$$

Also

$$\frac{\partial \varphi'}{\partial \theta} = \frac{e g}{r_0} \frac{\partial}{\partial \theta} \int (\psi_{1+} - \psi_{1-}) \, d p_\theta \qquad 3\text{-}5$$

Since the θ dependence of all the above quantities is $e^{il\theta}$ we obtain,

$$\begin{aligned} \dot{p}_{\theta+}^{ln} &= i l \frac{e v_+}{\sigma c} \left(e \frac{v_+}{c} \frac{\hat{g}}{r_0} \int \psi_+^{ln} d p_\theta - e \frac{v_-}{c} \frac{\hat{g}}{r_0} \int \psi_-^{ln} d p_\theta \right) \\ &\quad - i l \left(\frac{e g}{r_0} \int \psi_+^{ln} d p_\theta - \frac{e g}{r_0} \int \psi_-^{ln} d p_\theta \right) \qquad 3\text{-}6 \\ &= - i l e^2 \frac{g}{r_0} \int \left(\psi_+ - \psi_- + \frac{\hat{g}}{g} \frac{v_+ v_-}{\sigma c^2} \psi_- \right) d p_\theta \end{aligned}$$

We have written $\sigma = 1 + \frac{\nu \hat{g}}{\gamma^2} + \nu \frac{m_-}{m_+} \hat{g}$ and dropped one term because $\frac{v_+^2}{c^2} \ll 1$. Also some subscripts have been omitted.

For $\dot{p}_{\theta -}$ we obtain,

$$\begin{aligned} \dot{p}_{\theta-}^{ln} &= - i l \frac{e}{\sigma} \frac{v_-}{c} \left(e \frac{v_+}{c} \frac{\hat{g}}{r_0} \int \psi_+^{ln} d p_\theta - e \frac{v_-}{c} \frac{\hat{g}}{r_0} \int \psi_-^{ln} d p_\theta \right) \\ &\quad + e i l \left(\frac{e g}{r_0} \int \psi_+^{ln} d p_\theta - \frac{e g}{r_0} \int \psi_-^{ln} d p_\theta \right) \qquad 3\text{-}7 \\ &= i l \frac{e^2 g}{r_0} \int \left(\psi_+ - \psi_- + \frac{\hat{g}}{g} \frac{v_-^2}{\sigma c^2} \psi_- - \frac{\hat{g}}{g} \frac{v_+ v_-}{\sigma c^2} \psi_+ \right) d p_\theta \end{aligned}$$

Inserting the values of ψ_+ and ψ_- from eq. 1, we obtain from eqs. 6 and 7 after writing $\epsilon = \frac{- v_+ v_-}{\sigma c^2}$, $\frac{1}{\gamma_\sigma^2} = 1 - \frac{\hat{g}}{g} \frac{v_-^2}{\sigma c^2}$,

$$\dot{p}_{\theta +} = \frac{e^2 g}{r_0} \int \left(\frac{\dot{p}_{\theta +} \dfrac{\partial \psi_{0+}}{\partial p_0}}{\dot{\theta}_+ - \dfrac{\Omega}{2}} - \frac{(1+\epsilon)\, \dot{p}_{\theta -} \dfrac{\partial \psi_{0-}}{\partial p_0}}{\dot{\theta}_- - \dfrac{\Omega}{2}} \right) dp_0 \tag{3-8}$$

and

$$\dot{p}_{\theta -} = \frac{e^2 g}{r_0} \int \left(\frac{\dfrac{\dot{p}_{\theta -}}{r_v^2} \dfrac{\partial \psi_{0-}}{\partial p_0}}{\dot{\theta}_- - \dfrac{\Omega}{2}} - \frac{(1+\epsilon)\, \dot{p}_{\theta +} \dfrac{\partial \psi_{0+}}{\partial p_0}}{\dot{\theta}_+ - \dfrac{\Omega}{2}} \right) dp_0 \tag{3-9}$$

Note that $\dot{p}_{\theta\pm} \equiv \dot{p}_{\theta\pm}^{in}$ is a constant independent of p_θ .

To simplify we rewrite eqs. 8 and 9 and define

$$D_\pm \equiv \frac{e^2 g}{r_0} \int \frac{\dfrac{\partial \psi_{0\pm}}{\partial p_0}}{\dot{\theta}_\pm - \dfrac{\Omega}{2}}\, dp_0 \tag{3-10}$$

thus

$$\begin{aligned} \dot{p}_{\theta +} &= \dot{p}_{\theta +} D_+ - (1+\epsilon)\, \dot{p}_{\theta -} D_- \\ \dot{p}_{\theta -} &= -(1+\epsilon)\, \dot{p}_{\theta +} D_+ + \frac{1}{r_v^2}\, \dot{p}_{\theta -} D_- \end{aligned} \tag{3-11}$$

or

$$\begin{aligned} \dot{p}_{\theta +} (1 - D_+) &+ \dot{p}_{\theta -} (1+\epsilon) D_- = 0 \\ \dot{p}_{\theta +} (1+\epsilon) D_+ &+ \dot{p}_{\theta -} \left(1 - \frac{1}{r_v^2} D_-\right) = 0 \end{aligned} \tag{3-12}$$

which are two homogeneous equations in two unknowns. For a non-trivial solution the determinant of the coefficients of $\dot{p}_{\theta +}$, $\dot{p}_{\theta -}$ must vanish. Thus we obtain

$$\left|Det\right| = (1 - D_+)\left(1 - \frac{1}{\gamma_\sigma^2} D_-\right) - (1+\epsilon)^2 D_- D_+ = 0$$

or

$$1 - D_+ - \frac{1}{\gamma_\sigma^2} D_- + \left[\frac{1}{\gamma_\sigma^2} - (1+\epsilon)^2\right] D_+ D_- = 0 \qquad 3\text{-}13$$

In order to examine this dispersion relation more definitely, the terms in eq. 13 must be explicitly evaluated. By eq. 10,

$$D_- \equiv \frac{e^2 g}{r_o} \int \frac{\dfrac{\partial \psi_{o-}}{\partial p_o}}{\dot{\theta}_- - \dfrac{\Omega}{2}} \, dp_\theta \qquad 3\text{-}14$$

and by eq. D-2 and D-6, to first order in p_θ ,

$$\dot{\theta}_- = \dot{\theta}_{o-} + \frac{1}{\gamma m_o r_o^2}\left(-\frac{1}{1-n} + \frac{1}{\gamma^2}\right) p_o \qquad 3\text{-}15$$

$$= \dot{\theta}_{o-} - k\, p_o \qquad (k > 0)$$

Thus

$$D_- = \frac{e^2 g}{r_o} \int \frac{\dfrac{\partial \psi_{o-}}{\partial p_o}}{\dot{\theta}_{o-} - \dfrac{\Omega}{2} - k\, p_o} \, dp_o$$

Using now a pulse function for ψ, i.e. $\psi = \begin{cases} \dfrac{N}{2\pi\Delta} & p_o \le \dfrac{\Delta}{2} \\ 0 & p_o > \dfrac{\Delta}{2} \end{cases}$ we obtain

$$D_- = \frac{N e^2 g}{2\pi r_o} \frac{-k_-}{\left(\dot{\theta}_o - \dfrac{\Omega}{2}\right)^2 - \left(\dfrac{k_- \Delta_-}{2}\right)^2} \qquad \text{(See Appendix VII)} \qquad 3\text{-}16$$

It is clear from eq. 15 that $k_- \Delta_-$ is a measure of the velocity spread in the beam. It is therefore more instructive to write

$$\Delta \dot{\theta}_- = \frac{k_- \Delta_-}{2} = \frac{1\ \Delta_-}{2\gamma m_- r_0^2}\left(\frac{1}{1-n} - \frac{1}{\gamma^2}\right) = \frac{\alpha\ \Delta_-}{2\gamma m_- r^2} \qquad 3\text{-}17$$

or

$$r \Delta \dot{\theta}_- = \Delta v_- \qquad 3\text{-}18$$

and obtain

$$D_- = \frac{Ne^2 g}{2\pi r_0} \cdot \frac{c^2 \alpha}{\gamma m_- c^2 r_0^2} \cdot \frac{1}{(\Delta \dot{\theta})^2 - \left(\dot{\theta}_0 - \frac{\Omega}{\ell}\right)^2} \qquad 3\text{-}19$$

$$= \frac{c^2 \nu g \alpha}{(\Delta v_-)^2 - \left(v_- - r_0 \frac{\Omega}{\ell}\right)^2}$$

where $v_\pm = r\dot{\theta}_{0\pm}$ and $\nu = \frac{N}{2\pi r}\ \frac{e^2}{m_- c^2}$. The appropriate minus subscripts should be appended to all the appropriate quantities. The generalization to D+ is obvious. Here $\gamma = 1$, and we need only append the + subscripts. Thus, eq. 13 now becomes, after dividing through by $\nu \equiv \nu_-$

$$\frac{1}{\nu} = \frac{\frac{m_-}{m_+} c^2 \alpha_+ g}{(\Delta v_+)^2 - \left(v_+ - r_0 \frac{\Omega}{\ell}\right)^2} + \frac{\frac{1}{\gamma_\sigma^2}\ \frac{c^2}{\gamma} \alpha_- g}{(\Delta v_-)^2 - \left(v_- - r_0 \frac{\Omega}{\ell}\right)^2}$$

$$+ \frac{\frac{\nu}{\gamma}\ \frac{m_-}{m_+} c^4 \alpha_+ \alpha_- \left[(1+\epsilon)^2 - \frac{1}{\gamma_\sigma^2} g^2\right]}{\left[(\Delta v_+)^2 - \left(v_+ - r_0 \frac{\Omega}{\ell}\right)^2\right]\left[(\Delta v_-)^2 - \left(v_- - r_0 \frac{\Omega}{\ell}\right)^2\right]} \qquad 3\text{-}20$$

This equation is of fourth degree in $\frac{\Omega}{\ell}$. Since this is true even without the third term, that term will only modify the value of the roots resulting from the first two terms but not change their number. The third term may be neglected if ν is small enough as appears from the discussion below. A plot of this equation with the neglect of the third term on the R.H.S. of eq. 20 is drawn in Fig. 4, where $F\left(r_0 \frac{\Omega}{2}\right)$ denotes the two terms on the R.H.S. of eq. 20. The solid line denotes the region of validity of the equation, i.e., for $r_0 \frac{\Omega}{2}$ near v_+ or v_-. The dotted line would appear if the equation were taken seriously in the rest of the domain of $r_0 \frac{\Omega}{2}$.

Under the assumptions used in deriving eq. 20, we will show that it may be written as two separate equations. There are four roots, of which one pair is near v_+, and the other near v_-. These may be complex. Since the beams are in a magnetic field at the same position and $\dot{\theta} = \frac{eB}{m c}$ holds, we have $\gamma m_- c = m_+ v_+$. As long as the ions are far from being relativistic, $v_- \gg v_+$ will hold. Consider this case and the pair of roots near v_-. Eq. 20 then becomes

$$\frac{1}{\nu} = -\frac{c^2}{v_-^2}\frac{\alpha_- g}{\gamma_v^2 r} + \frac{\frac{m_-}{m_+} c^2 \alpha_+ g}{(\Delta v_+)^2 - \left(v_+ - r_0 \frac{\Omega}{2}\right)^2}\left(1 - \frac{\nu \alpha_- g}{r \sigma}\right) \qquad 3\text{-}21$$

Two more terms in this equation may be neglected. In deriving the constraint equation the magnetic field due to the beam current was neglected. This means that $\frac{\nu g}{r} \ll 1$. Since $\gamma m_- c = m_+ v_+$ this implies also $\nu g \frac{m_-}{m_+} \ll 1$, and hence $\sigma \approx 1$ and we can neglect the third term on the R.H.S. of eq. 21. Also since $v_- \simeq c$, the relation $\frac{\nu g}{r} \ll 1$ implies that the first erm on the R.H.S. of eq. 21 is much less than $\frac{1}{\nu}$ and may also be neglected. Thus we obtain

$$\left(r_0 \frac{\Omega}{2} - v_+\right)^2 = (\Delta v_+)^2 - \nu \frac{m_-}{m_+} c^2 \alpha_+ g \qquad 3\text{-}22$$

which is the same equation as the N. M. I. dispersion relation for the proton steam alone.

For the pair of roots of eq. 20 near V_- , the equation may be similarly reduced. Setting $r_0 \frac{\Omega}{l} \approx V_-$ in eq. 20 we obtain

$$\frac{1}{\nu} = -\frac{m_-}{m_+}\frac{c^2}{V_-^2}\alpha_+ g + \frac{\frac{c^2\alpha_- g}{\gamma^2}\left(1 - \frac{\hat{g}}{g}\frac{V_-^2}{\sigma c^2} - \nu\frac{m_-}{m_+}\alpha_+ g\right)}{(\Delta V_-)^2 - \left(V_- - r_0\frac{\Omega}{l}\right)^2} \tag{3-23}$$

Since $\nu g \frac{m_-}{m_+} \ll 1$, we may again neglect the first term on the R. H. S. of eq. 23. The term in parenthesis may be simplified. If $\gamma^3 m_- \gg m_+$ then $\frac{1}{\sigma} \equiv \frac{1}{1 + \frac{\nu\hat{g}}{\gamma^3} + \nu\frac{m_-}{m_+}\hat{g}} \approx 1 - \nu\hat{g}\frac{m_-}{m_+}$.
Recalling now that $\alpha_+ = \frac{1}{1-h} - 1$ we may write

$$1 - \frac{\hat{g}}{g}\frac{V_-^2}{\sigma c^2} - \nu\frac{m_-}{m_+}\alpha_+ g \approx \frac{1}{\gamma^2} - \frac{V_-^2}{c^2}\frac{\delta g}{g} - \nu\frac{m_-}{m_+}g\left(\frac{1}{1-h} - 2\right)$$

so that eq. 23 becomes

$$\frac{1}{\nu} = \frac{\frac{c^2\alpha_- g}{\gamma}\left[\frac{1}{\gamma_g^2} - \nu\frac{m_-}{m_+}g\left(\frac{1}{1-h} - 2\right)\right]}{(\Delta V_-)^2 - \left(V_- - r_0\frac{\Omega}{l}\right)^2} \tag{3-24}$$

This equation differs from the N. M. I. equation for the relativistic electrons alone by an additional term in the bracket, which contributes to stability if $h > \frac{1}{2}$. If the bracket term is positive then there is stability even if $\Delta V_- = 0$.

Eqs. 22 and 24 must be consistent with the inequalities mentioned earlier which are required for the validity of these equations. The validity of the constraint equation requires that $\left|V_\pm - r_0\frac{\Omega}{l}\right| \ll \frac{\sqrt{1-h}}{l} V_\pm$ by eq. 2-7. By the definition of $\Delta V_\pm$ in eqs. 17 and 18, and the constraint eq. 2-6, because the beam is thin, one can show easily that also $\Delta V_{\pm} \ll V_{\pm}$. These conditions are satisfied for the negative stream if $\frac{\nu g}{\gamma} \ll 1$ and $\nu g \frac{m_-}{m_+} \ll 1$. For the positive stream using $\gamma m_- c = m_+ V_+$, the restriction is stronger, $\frac{\nu g}{\gamma} \ll \frac{\gamma m_-}{m_+}$ is required.

Summarizing, we find that the N.M.I. dispersion relation for two streams, is given by eqs. 22 and 24 when

$$\Delta V_{\pm} \ll V_{\pm} \,, \quad \frac{\nu g}{\gamma} \ll \frac{\gamma m_-}{m_+} \,, \quad \gamma m_- V_- = m_+ V_+ \,, \quad \frac{\gamma^3 m_-}{m_+} \gg 1 \qquad 3\text{-}25$$

are satisfied and the electrons are relativistic and the protons are not. Eq. 22 is the same as the N.M.I. equation obtained if only the protons were present, while the single stream N.M.I. equation for the electrons is modified by the presence of the protons. If the electrons are also non-relativistic then eq. 24 reduces to eq. 22 where the + subscripts are replaced by - and the validity of the equations is given by $\Delta V_{\pm} \ll V_{\pm}$, $\nu g \ll \frac{m_-}{m_+} \frac{V_-^2}{c^2}$, and $m_- V_- = m_+ V_+$.

Sec. 3.2 - Longitudinal Oscillations in Thin Beams

a) Restricted equation

The dispersion relation for longitudinal oscillations of thin beams will be derived below as a limiting case of the two-stream N.M.I. equations. Since the p terms in eqs. 6 and 7 were neglected, this dispersion relation is valid only for cool beams, and for values of $r_0 \frac{\Omega}{l}$ near V_+ or V_-. This defect, equivalent to a limitation on ν, will be removed later in another derivation, where the p_θ terms are retained.

Longitudinal oscillations describe a motion in which the particles move in a line along the average beam velocity without any transverse motion. In the N.M.I. situation the transverse motion is described by the constraint equation, $\delta r \equiv r - r_o = -\frac{1}{1-h}\frac{c}{eA_o}p_\theta$ and hence if $\frac{1}{1-h} \to 0$, $\delta r \to 0$ and there is no transverse motion. This means that an infinitely strong focusing field inhibits the transverse motion. Therefore, one need only let $\frac{1}{1-h} \to 0$ in eq. 20 to obtain the dispersion relation for longitudinal oscillations. Hence there exists only one dispersion relation for beams in a magnetic field and it depends on the value of $1-h$. Since, in general $1-h$ is small and finite, the examination of the instabilities of contra-streaming particles[(5), (7)] using the thin beam longitudinal dispersion relation is equivalent to assuming $\frac{1}{1-h} = 0$, which is never legitimate. It is, however, of interest to derive this relation because first, it will be valid for accelerators with strong focusing, of such strength that it may be considered infinite, and secondly, as will be shown later, this relation is valid for thin linear beams, and has been derived elsewhere[(5)] enabling the consistency of our treatment to be checked.

Let now $\frac{1}{1-h} \to 0$ in eq. 20. Since only $\alpha_\pm$ (defined in eq. 17) depends on $1-h$, one obtains the result that

$$\alpha_+ \to -1 \; , \; \alpha_- \to -\frac{1}{\gamma^2} \quad \text{when} \quad \frac{1}{1-h} \to 0 \qquad \text{3-26}$$

For this case too, we will show that eq. 20 reduces to eqs. 22 and 24. Eq. 20 is valid when the inequalities

$$\left|v_+ - r_0\frac{\Omega}{l}\right| \ll \left|\alpha_+ c\right| = c\,, \quad \left|v_- - r_0\frac{\Omega}{l}\right| \ll \left|\alpha_- c\right| = \frac{c}{\gamma^2} \qquad \text{3-27}$$

hold, because then by the discussion at the end of Sec. 2.2, the p_θ terms under the integral sign of eq. 3, the expression for A'_θ , may be neglected. The neglect of the p_θ terms outside the integral sign, which arise from v_θ in eq. 2, implies that $\Delta v_+ \ll c$, when the ions are non-relativistic and $\Delta v_- \ll \frac{c}{\gamma^2}$, as also explained at the end of Sec. 2.2. Applied to eq. 22, these inequalities require that $\nu g \frac{m_-}{m_+} \ll 1$ When applied to eq. 24, the inequality

$$\nu g \gamma \left[\frac{1}{\gamma_g^2} + 2\nu \frac{m_-}{m_+} g\right] \ll 1$$

results. We note further that to obtain eqs. 22 and 24 from eqs. 21 and 23, requires that $\frac{\nu g}{\gamma^3} \ll 1$, besides $\nu g \frac{m_-}{m_+} \ll 1$. We shall also assume that $\gamma^3 m_- \gg m_+$ and therefore $\frac{1}{\sigma} \approx 1 - \nu g \frac{m_-}{m_+}$. When all these assumptions hold, eqs. 22 and 24 result where $\alpha_\pm$ are given by eq. 26.

In deriving the constraint equation, it was required that $\left|v_\pm - r_0\frac{\Omega}{l}\right| \ll \frac{\sqrt{1-h}}{l} v_\pm$. Since here $\frac{1}{1-h} \to 0$, this inequality is always true and poses no restriction on the parameters.

The longitudinal dispersion relation for thin beams may be put into a more instructive form by combining eqs. 22 and 24 (with the substitutions eq. 26) to give

$$\frac{1}{\nu} = \frac{\frac{m_-}{m_+} c^2 g}{\left(v_+ - r_0\frac{\Omega}{l}\right)^2 - (\Delta v_+)^2} + \frac{\frac{c^2 g}{\gamma^3}\left(\frac{1}{\gamma_g^2} + 2\nu\frac{m_-}{m_+} g\right)}{\left(v_- - r_0\frac{\Omega}{l}\right)^2 - (\Delta v_-)^2} \qquad \text{3-28}$$

which is valid when $\Delta v_+ \ll c$, $v_+ \ll c$, $\Delta v_- \ll \frac{c}{\gamma^2}$, $\frac{\nu g}{\gamma^3} \ll 1$, $\nu g \frac{m_-}{m_+} \ll 1$ and $\nu g \gamma \left[\frac{1}{\gamma_g^2} + 2\nu \frac{m_-}{m_+} g\right] \ll 1$. We record again the definitions,

$$\Delta v_- = \frac{\Delta_-}{2\gamma^3 m_- r_0}, \quad \Delta v_+ = \frac{\Delta_+}{2 m_+ r_0}, \quad \frac{1}{\gamma_g^2} = \frac{1}{\gamma^2} - \frac{v_-^2}{c^2}\frac{\delta g}{g}$$

For positive numerators, the two terms on the R.H.S. of this equation are plotted in Fig. 5. The domain of validity is indicated schematically by the solid line.

This equation differs from the one derived by Finkelstein and Sturrock,[4] by the inclusion of temperature terms, but more important by the additional factor $\frac{1}{\gamma^2} - \frac{v_-^2}{c^2}\frac{\delta g}{g} + 2\nu \frac{m_-}{m_+} g$, multiplying the second term on the R.H.S. of the equation. For small l , and large γ , the $\frac{\delta g}{g}$ term can be large enough so that this factor is negative. This will give an instability if Δv_- is small enough, as is evident from Fig. 4. Thus, in strong focusing machines below transition, where the above approximations apply, the lowest wave numbers modes will grow. This is a new instability.

For a linear geometry which F-S consider, $\delta g \equiv 0$ as is evident from Appendix III. Thus when $\delta g = 0$, and ν is very small, this additional factor multiplying the second term on the R.H.S. of eq. 28 becomes $\frac{1}{\gamma^2}$. As $\nu g \frac{m_-}{m_+} \to 1$, the factor approaches 2, although when $\nu g \frac{m_-}{m_+} = 1$, the equation breaks down. Thus for very small ν this difference means that the stable longitudinal oscillations near v_- are given by eq. 28 as

$$r_0 \frac{\Omega}{l} = v_- \pm c \left[\frac{\nu g}{\gamma^5} + \left(\frac{\Delta v}{c}\right)^2\right] \qquad \text{3-29}$$

The F-S equation has instead the factor r^3 instead of r^5. Also their $g = 2 \ln \frac{1}{k_z \rho}$, which differs slightly from our g given by eqs. C-16 and C-13 because the geometries differ. We cannot compare stability criteria with F-S because eq. 28 is only valid for small ν.

When eq. 28 is applied to the longitudinal oscillations of thin linear streams that are very long, it is necessary also to replace $r_0 \frac{\Omega}{l}$ by $\frac{\Omega}{k_z}$, where any beam distrubance is represented by $\varphi = \varphi_{k\Omega} e^{ik_z z - i\Omega t}$. This is true from the following observations. If $\frac{1}{1-h} = 0$, $\delta r \equiv r - r_0 = 0$. Hence $r = r_0$, a constant, in eqs. A-2b and A-3b, and these equations correspond exactly to eqs. A-2c and A-3c if the substitutions $r_0 \theta = z$, $r_0 \dot{\theta} = \dot{z}$, $\frac{p_\theta}{r_0} = p_z$, $\frac{\dot{p}_\theta}{r_0} = \dot{p}_z$, $A_{\theta_0} = A_z$ are made. Thus eq. 28 is valid either for devices of circular symmetry with infinitely strong focusing or for linear beams without magnetic fiel ds.

b) Exact equation

Now we present a derivation of the longitudinal dispersion relation where the p_θ terms of eq. 6, 7 are retained. (We shall also assume that the beam is sufficiently thin, or that the geometry is linear, so that $\hat{g} = g$ and $\delta g = 0$.) These two equations are now, with the substitution of $\psi_{\pm}^{l\Omega}$ from eq. 1,

$$\dot{p}_{\theta_+}^{l\Omega}$$

$$= \frac{e^2 g}{r_0} \left[- \left(\frac{v_+}{c} + \frac{p_\theta}{c\, m_+ r_0} \right) \frac{1}{\sigma} \int \left(\frac{v_+}{c} + \frac{p_\theta}{c\, m_+ r_0} \right) \frac{\dot{p}_{\theta_+}^{l\Omega} \frac{\partial \psi_{0+}}{\partial p_\theta}}{\dot{\theta}_+ - \frac{\Omega}{l}} dp_\theta + \int \frac{\dot{p}_{\theta_+}^{l\Omega} \frac{\partial \psi_{0\pm}}{\partial p_\theta}}{\dot{\theta}_+ - \frac{\Omega}{l}} dp_\theta \right. \tag{3-30}$$

$$+\left(\frac{V_+}{c}+\frac{p_\theta}{c\,m_+ r_0}\right)\frac{1}{\sigma}\int\left(\frac{V_-}{c}+\frac{p_\theta}{c\,m_-\gamma^3 r_0}\right)\frac{\dot{p}_{\theta_-}^{2\Omega}\dfrac{\partial\psi_{0-}}{\partial p_\theta}}{\dot{\theta}_- - \dfrac{\Omega}{2}}\,dp_\theta-\int\frac{\dot{p}_{\theta_-}^{2\Omega}\dfrac{\partial\psi_{0-}}{\partial p_\theta}}{\dot{\theta}_- - \dfrac{\Omega}{2}}\,dp_\theta\Bigg]$$

$$\dot{p}_{\theta_-}^{2\Omega}$$

$$=\frac{e^2 q}{r_0}\Bigg[-\left(\frac{V_-}{c}+\frac{p_\theta}{c\,m_-\gamma^3 r_0}\right)\frac{1}{\sigma}\int\left(\frac{V_-}{c}+\frac{p_\theta}{c\,m_-\gamma^3 r_0}\right)\frac{\dot{p}_{\theta_-}^{2\Omega}\dfrac{\partial\psi_{0-}}{\partial p_\theta}}{\dot{\theta}_- - \dfrac{\Omega}{2}}\,dp_\theta+\int\frac{\dot{p}_{\theta_-}^{2\Omega}\dfrac{\partial\psi_{0-}}{\partial p_\theta}}{\dot{\theta}_- - \dfrac{\Omega}{2}}\,dp_\theta \qquad \text{3-31}$$

$$+\left(\frac{V_-}{c}+\frac{p_\theta}{c\,m_-\gamma^3 r_0}\right)\frac{1}{\sigma}\int\left(\frac{V_+}{c}+\frac{p_\theta}{c\,m_+ r_0}\right)\frac{\dot{p}_{\theta_+}^{2\Omega}\dfrac{\partial\psi_{0+}}{\partial p_\theta}}{\dot{\theta}_+ - \dfrac{\Omega}{2}}\,dp_\theta-\int\frac{\dot{p}_{\theta_+}^{2\Omega}\dfrac{\partial\psi_{0+}}{\partial p_\theta}}{\dot{\theta}_+ - \dfrac{\Omega}{2}}\,dp_\theta\Bigg]$$

Since eqs. 30 and 31 are linear in p_θ the simplest assumption for $\dot{p}_\theta^{2\Omega}$ (the unknown quantities) which gives a solution for eqs. 30 and 31, is

$$\dot{p}_{\theta\pm}^{2\Omega} = a_\pm + b_\pm p_\theta$$

If we substitute this into the equations, each equation is still linear in p_θ. Let all the terms in eq. 30 and 31 be put on the left side of the equality so that they equal zero. Since the coefficient of the constant and the coefficient of p_θ must both be zero, each equation now gives two equations. There are now four equations homogenous in the four unknowns $a_\pm$, $b_\pm$. By setting the determinant of the coefficient equal to zero, we obtain a 4 x 4 determinant which is the exact dispersion relation.

There are two simple relations which allow the terms to be simplified after the formidable number of terms of the determinant are obtained. Note first that $\dot{\theta}_\pm = \dot{\theta}_{0\pm} - k_\pm p_0$. Then the integrals which appear in the determinant are of the two forms

$$\int \frac{p \frac{\partial \psi_0}{\partial p} \, dp}{\dot{\theta}_0 - \frac{\Omega}{2} - kp} \quad , \quad \int \frac{p^2 \frac{\partial \psi_0}{\partial p} \, dp}{\dot{\theta}_0 - \frac{\Omega}{2} - kp}$$

Let $\dot{\theta}_0 - \frac{\Omega}{2} \equiv \lambda$ to simplify the algebra. Then, since $\int \frac{\partial \psi}{\partial p} dp = 0$

$$\int \frac{p \frac{\partial \psi}{\partial p} \, dp}{\lambda - kp} = \int \frac{p \frac{\partial \psi}{\partial p}}{\lambda - kp} dp + \frac{1}{k} \int \frac{\partial \psi}{\partial p} dp$$

$$= \int \frac{\left(p + \frac{\lambda}{k} - p\right) \frac{\partial \psi}{\partial p}}{\lambda - kp} dp \qquad \text{3-32}$$

$$= \frac{\lambda}{k} \int \frac{\frac{\partial \psi}{\partial p}}{\lambda - kp} dp$$

gives the first desired relation.

To obtain the second relation, note first that

$$\int p \frac{\partial \psi}{\partial p} dp = p \psi \Big|_{-\infty}^{\infty} - \int \psi \, dp = - \frac{N}{2\pi}$$

because ψ_0 is zero at the limits $\pm \infty$, and is normalized to $\frac{N}{2\pi}$.

The second desired relation is now

$$\int \frac{p^2 \frac{\partial \psi}{\partial p} dp}{\lambda - kp} = \int \frac{p^2 \frac{\partial \psi}{\partial p}}{\lambda - kp} dp + \frac{1}{k} \int p \frac{\partial \psi_0}{\partial p} dp + \frac{N}{2\pi}$$

$$= \int \frac{\left(p^2 + \lambda \frac{p}{k} - p^2\right) \frac{\partial \psi}{\partial p}}{\lambda - kp} dp + \frac{N}{k \cdot 2\pi} \qquad \text{3-33}$$

$$= \frac{\lambda}{k} \int \frac{p \frac{\partial \psi}{\partial p}}{\lambda - kp} dp + \frac{N}{k \cdot 2\pi}$$

$$= \left(\frac{\lambda}{k}\right)^2 \int \frac{\frac{\partial \psi}{\partial p}}{\lambda - kp} dp + \frac{N}{k \cdot 2\pi}$$

where we have also used eq. 32. $\left(\lambda_\pm \equiv \dot{\theta}_{0\pm} - \frac{\Omega}{2}\right)$.

Besides these two integral relations, the following relations are needed too,

$$\frac{e^2 g}{r_0}\cdot\frac{1}{(c\gamma^3 m_- r_0)^2}\cdot\frac{N}{2\pi k_-} = \frac{e^2 g}{r_0}\cdot\frac{1}{(c\gamma^3 m_- r_0)^2}\cdot\frac{N}{2\pi\left(\frac{\alpha_-}{\gamma m_- r_0^2}\right)} = \frac{\nu g}{\gamma^3}\cdot\frac{1}{\alpha_-\gamma^2}$$

and

$$\frac{e^2 g}{r_0}\cdot\frac{1}{(c\, m_+ r_0)^2}\cdot\frac{N}{2\pi k_+} = \frac{\nu\, m_-}{m_+}\, g\cdot\frac{1}{\alpha_+} \qquad \text{3-34}$$

Till now all the equations are equally valid for the N. M. I. For the longitudinal oscillations one must let $\frac{1}{1-n} \to 0$ in eqs. 32, 33 and 34 so that

$$\alpha_+ = -1,\ \alpha_- = -\frac{1}{\gamma^2} \text{ and } k_+ = \frac{\alpha_+}{m_+ r_0^2} = \frac{-1}{m_+ r_0^2},\quad k_- = \frac{\alpha_-}{\gamma m_- r_0^2} = \frac{-1}{\gamma^3 m_- r_0^2}$$

Many terms cancel so that the dispersion relation has finally the simple form,

$$\frac{1}{1-\left(\frac{\Omega r_0}{2c}\right)^2} = \frac{e^2 g}{r_0}\int\frac{\frac{\partial \psi_{0+}}{\partial p_0}\, dp_0}{\left(\dot{\theta}_{0+}-\frac{\Omega}{2}\right)+\frac{p_0}{m_+ r_0^2}} + \frac{e^2 g}{r_0}\int\frac{\frac{\partial \psi_{0-}}{\partial p_0}\, dp_0}{\left(\dot{\theta}_{0-}-\frac{\Omega}{2}\right)+\frac{p_0}{\gamma^3 m_- r_0^2}} \qquad \text{3-35}$$

If we use a pulse function for ψ_0, as given by eq. G-1, to evaluate the integrals, eq. 35 becomes

$$\frac{1}{\nu} = \frac{\left[1-\left(\frac{\Omega r_0}{2c}\right)^2\right]\frac{m_-}{m_+}c^2 g}{\left(V_+ - r_0\frac{\Omega}{2}\right)^2 - \left(\Delta V_{\parallel +}\right)^2} + \frac{\left[1-\left(\frac{\Omega r_0}{2c}\right)^2\right]\frac{c^2 g}{\gamma^3}}{\left(V_- - r_0\frac{\Omega}{2}\right)^2 - \left(\Delta V_{\parallel -}\right)^2} \qquad \text{3-36}$$

where $\Delta V_{\parallel +} = \frac{\Delta_+}{2 m_+ r_0}$, $\Delta V_{\parallel -} = \frac{\Delta_-}{2\gamma^3 m_- r_0}$ and there is now no restriction on the ν values allowed in this equation. There are, however, restrictions on p_0 if eq. 36 is valid as written. Since none of the protons are relativistic $\gamma_+^2 \approx 1$ and

$$\frac{1}{1-\frac{V_+^2}{c^2}} = \frac{1}{1-\left(\frac{V_+ + \Delta V_+}{c}\right)^2} = \frac{1}{1-\left(\frac{V_+}{c} + \frac{\Delta p_+}{c\, m_+ r}\right)^2} \approx 1$$

implies that if $v_+ \ll c$ that $\frac{\Delta p_+}{m_+ r_0} \ll c$. Also if all of the electrons are associated with one value of γ , then

$$\gamma_-^2 = \frac{1}{1-\frac{V_-^2}{c^2}} = \frac{1}{1-\left(\frac{V_-}{c} + \frac{\Delta p}{c\,\gamma^2 m_- r_0}\right)^2} = \frac{1}{\frac{1}{\gamma^2}\left(1 - \frac{\Delta p}{c\,\gamma\, m_- r_0} - \frac{1}{\gamma^3}\left(\frac{\Delta p}{c\,\gamma\, m_- r}\right)^2\right)}$$

and it is required that $\frac{\Delta p_-}{\gamma\, m_- r_0} \ll c$. These limitations apply also to p_θ in eq. 35, because the validity of eq. D-2 requires p_θ to be small.

It is easily verified that in the domain where eqs. 28 and 36 are valid the equations give the same result. The term $\frac{1}{\gamma^2} + 2\nu \frac{m_-}{m_+} g$ becomes $1 - \left(\frac{\Omega r_0}{l c}\right)^2$ in eq. 36. The criterion for stability may now be derived from eq. 36. As is evident from Fig. 5, one need only find the minimum of the R.H.S. of eq. 36 and ensure that it is less than $\frac{1}{\nu}$. The result is that for $\gamma^3 m_- \gg m_+$, $\nu g < \frac{m_+}{m_-} \frac{1}{4} \left(\frac{\gamma^3 m_-}{m_+}\right)^{1/2}$ F-S give the criteria for stability, when $\gamma^3 m_- \gg m_+$, as $\nu g < \frac{m_+}{m_-}$ for their equation which does not contain the factors $1 - \left(\frac{\Omega r_0}{l c}\right)^2$. Thus eq. 36 gives an improvement in the stability criterion for large γ .

Sec. 3.3 - Longitudinal Oscillations in Infinitely Wide Beams

With the formalism developed above, it is now a simple matter to find the dispersion relation for the case of infinitely wide beams, or beams with a perturbation wavelength $\frac{r_0}{l}$ or $\frac{2\pi}{k_z}$, much less than the beam width. Assume that the beams travel in the z direction, and that all quantities E_z, φ, A_z have the z, t dependence given by $e^{ik_z z - i\Omega t}$. From Maxwell's equations

$$\nabla \cdot E = 4\pi q$$

or 3-37

$$i k_z E_k = 4\pi q_k$$

Also

$$E = -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}$$

and 3-38

$$E_k = -i k_z \varphi_k + i\frac{\Omega}{c} A_k$$

The gauge condition gives

$$\nabla \cdot A + \frac{1}{c}\dot{\varphi} = 0$$

or 3-39

$$i k_z A_k - i\frac{\Omega}{c}\varphi_k = 0$$

Combining now eq. 37, 38 and 39, we obtain

$$\varphi_k = 4\pi q_k \frac{1}{k_z^2} \frac{1}{1 - \left(\frac{\Omega}{k_z c}\right)^2}$$

Comparing this expression with eq. 5, we obtain

$$q_k = \frac{1}{\pi\rho^2}\frac{1}{r_0} e \int (\psi_{1+} - \psi_{1-})\, d\rho_0 \tag{3-40}$$

as the charge per unit volume. Thus in eq. 5 one may set

$$g = 4\pi \cdot \frac{1}{\pi\rho^2 k_z^2} \cdot \frac{1}{1 - \left(\frac{\Omega}{k_z c}\right)^2} \tag{3-41}$$

From eq. 39,

$$A_k = \frac{\Omega}{k_z c}\varphi_k = \frac{r_0\,\Omega}{l\,c}\varphi_k \tag{3-42}$$

Using eq. 4, with the p_θ terms included and eqs. 32 and 33, eq. 42 is satisfied if the terms $\frac{\nu g}{\gamma^3}$, $\nu \frac{m_-}{m_+} g = 0$ and the term $\frac{N}{k \cdot 2\pi}$ in eq. 33 is set equal to zero. The dispersion relation is now obtained from eq. 30 and 31, with the proviso that the three mentioned terms are zero, and g is given by eq. 41. Following the same procedure as outlined in the above section and solving the 4 x 4 determinant, eq. 35 is again obtained for the dispersion relation. Using the value of g given in eq. 41, the factor $1 - \left(\frac{\Omega r_0}{\ell c}\right)^2$ is now cancelled and we obtain

$$1 = \frac{4\pi e^2}{\pi \rho^2 k_z^2} \frac{N}{2\pi r_0} \left[\int \frac{\frac{\partial \psi'_{0+}}{\partial p_0} \, dp_0}{\dot{\theta}_{0+} - \frac{\Omega}{\ell} + \frac{p_0}{m_+ r_0^2}} + \int \frac{\frac{\partial \psi'_{0-}}{\partial p_\theta} \, dp_\theta}{\dot{\theta}_{0-} - \frac{\Omega}{\ell} + \frac{p_0}{\gamma^3 m_- r_0^2}} \right] \qquad 3\text{-}43$$

ψ'_0 is now normalized to 1, on the field p_θ , since the $\frac{N}{2\pi}$ factor has been factored out. Since $\frac{N}{2\pi r_0 \cdot \pi \rho^2} = n$ = density of particles in the toroidal geometry, eq. 43 can also apply to a linear geometry, where 'n' is the density. Also one may write $\frac{dp_0}{r_0} = dp_z$, and normalize ψ'_0 to 1 on the field p_z so that each integral becomes

$$I = \int \frac{\frac{\partial \psi_0}{\partial p_z} \, dp_z}{\left(V - r_0 \frac{\Omega}{\ell}\right) + \frac{p_z}{M}}$$

where $M_+ = m_+$, $M_- = \gamma^3 m_-$. Making now a final change of variable and letting $\frac{p_z}{M} = V_z$, and normalizing ψ_0 to 1 on the field V_z , and letting $\frac{r_0}{\ell} = \frac{1}{k_z}$, eq. 43 becomes

$$k_z^2 = \omega_{p+}^2 \int \frac{\frac{\partial \psi_{0+}}{\partial V_z} \, dV_z}{\left(V_+ - \frac{\Omega}{k_z}\right) + V_z} + \frac{\omega_{p-}^2}{\gamma^3} \int \frac{\frac{\partial \psi_{0-}}{\partial V_z} \, dV_z}{\left(V_- - \frac{\Omega}{k_z}\right) + V_z} \qquad 3\text{-}44$$

where $\omega_{p\pm}^2 = \frac{4\pi n_\pm e^2}{m_\pm}$ and is valid when $\Delta p_+ \ll m_+ c r,\ \Delta p_- \ll \gamma m_- c r$. Note that the actual velocity of a particle V is given by $V_\pm = v_\pm + V_\pm$.

If the positive beam is at zero temperature and moving with zero velocity, then the first integral in eq. 44 reduces to $\frac{k_\pm^2}{\Omega^2}$ and eq. 44 is identical to a result of Bludman, et al.(18) as shown in Appendix X.

The reason for the additional factor $1-\left(\frac{\Omega r_0}{\ell c}\right)^2 \equiv \frac{1}{\gamma_{\Omega\ell}^2}$ that appears in eq. 35 for thin beams, but not for infinite beams, as in eq. 44, may now be noted. It arises because the force between two small elements of a thin tube, far apart, moving with velocity $\frac{\Omega r_0}{\ell}$, is decreased by the factor $\gamma_{\Omega\ell}^2$, because the electric field is decreased by this amount. The decrease arises because the force is like that between two small charges and is $\sim \frac{1}{d^2}$. The longitudinal electric field is invariant, hence the field is obtained by writing $E = \frac{e}{d^2}$, where d is measured in the rest system of the charges. Since in the lab system this distance is observed contracted, i.e., $\frac{d^2}{\gamma_{\Omega\ell}^2} = d_{lab}^2$, we obtain $E = \frac{e}{\gamma_{\Omega\ell}^2 d_{lab}^2}$. The wavelength of the perturbation $\lambda = \frac{r_0}{\ell}$, corresponds to d_{lab}. The forces and the electric fields are the same whether there are actual moving charge clumps or a nearly stationary charge fluid where the clumps appear to move due to the phase velocity of the disturbance because the charge density, only, appears in Maxwell's equations. Thus the factor $\gamma_{\Omega\ell}^2$ appears for the thin beam. It is also possible to show from the formalism of eqs. 30 and 31 and the equations following, using $E = -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}$ that the longitudinal field does in fact have this $\gamma_{\Omega\ell}^2$ factor for the thin beam case.

In the infinitely wide beam situation, the elements are two plane sheets, the electric field is $E = 2\pi\sigma$, and does not depend on $r_{\Omega 1}$, nor does the force, so that the factor $r_{\Omega 1}^2$ does not appear.

If only one beam is present, then it is possible to check our additional factor $1 - \left(\frac{\Omega r_0}{2c}\right)^2$, by making a Lorentz transformation from a stationary 'beam' to a moving one. To simplify the algebra, the beam is assumed cold. We shall assume the ion beam to be absent. For small currents, i.e., $\frac{\nu g}{\gamma} \ll 1$, $1 - \left(\frac{\Omega r_0}{2c}\right)^2 \rightarrow \frac{1}{\gamma^2}$. The resulting equations derived from eq. 36 and 44 are given in Appendix XI as eqs. K-8 and K-1 respectively. The details and the Lorentz transformations are done in Appendix XI and the equations are consistent. It is suggested that the neglect of the retardation terms adds a term $2 \ln \frac{1}{\sqrt{1 - \left(\frac{\Omega r_0}{2c}\right)^2}}$ to g as defined in eqs. C-13 and C-16, at least in the linear case.

CHAPTER IV - SUMMARY

The dispersion relations are summarized here (see p. 123 for definitions) :

A. SINGLE STREAM N.M.I. (eq. 2-39)

$$1 = \frac{e^2 g}{r_0 \gamma_g^2} \int \frac{\frac{\partial \psi_0}{\partial p_\theta} \, dp_\theta}{\dot{\theta}_0 - \frac{\Omega}{l} - k p_\theta} \qquad \mathrm{Im}\,\Omega > 0$$

$$1 = \frac{e^2 g}{r_0 \gamma_g^2} \int \frac{\frac{\partial \psi_0}{\partial p_0} \, dp_0}{\dot{\theta}_0 - \frac{\Omega}{l} - k p_0} - 2\pi i \frac{e^2 g}{k r_0 \gamma_g^2} \left. \frac{\partial \psi_0}{\partial p_0} \right)_{p_0 = \left(-\frac{\Omega}{l} + \dot{\theta}_0\right)\frac{1}{k}} \qquad \mathrm{Im}\,\Omega < 0 \tag{4-1}$$

Initial velocity distributions investigated:

1) Pulse function

$$\psi_0 = \begin{cases} \dfrac{N}{2\pi\Delta} & |p_0| \leq \dfrac{\Delta}{2} \\ 0 & |p_0| > \dfrac{\Delta}{2} \end{cases} \tag{4-2}$$

Dispersion relation (eq. G-3)

$$\frac{\Omega}{l} - \dot{\theta}_0 = \frac{\pm \alpha}{\gamma m_0 r_0^2} \left[\frac{\Delta^2}{4} - \frac{\nu g \gamma}{\alpha \gamma_g^2} (m_0 c r_0)^2 \right]^{1/2}$$

Stability criteria

$$\frac{1}{\gamma_g^2} < 0 \quad \text{or if} \quad \frac{1}{\gamma_g^2} > 0 \quad \text{then} \quad \frac{\Delta}{2} > \sqrt{\frac{\nu g \gamma}{\alpha \gamma_g^2}} \, m_0 c r_0$$

2) Resonance function

$$\psi_0 = \frac{N}{2\pi} \cdot \frac{\Delta}{\pi} \frac{1}{p_0^2 + \Delta^2}$$

Dispersion relation (eq. F-15)

$$\frac{\Omega}{l} - \dot{\theta}_0 = i \frac{\alpha}{\gamma m_0 r_0^2} \left[-\Delta \pm \frac{\nu g \gamma}{\alpha \gamma_g^2} (m_0 c r_0) \right] \tag{4-3}$$

Stability criteria

$$\frac{1}{\gamma_g^2} < 0 \quad \text{or} \quad \Delta > \sqrt{\frac{\nu g \gamma}{\alpha\, \gamma_g^2}}\; m_o c r_o \quad \text{if} \quad \frac{1}{\gamma_g^2} > 0$$

3) Maxwellian distribution

$$\psi_o = \frac{N}{2\pi}\,\frac{1}{\Delta\sqrt{\pi}}\, e^{-\frac{p_o^2}{\Delta^2}}$$

Stability criteria (eq. E-17, E-18)

4-4

$$\frac{1}{\gamma_g^2} < 0 \quad \text{or} \quad \frac{\Delta}{\sqrt{2}} > \sqrt{\frac{\nu g \gamma}{\alpha\, \gamma_g^2}}\; m_o c r_o \quad \text{if} \quad \frac{1}{\gamma_g^2} > 0$$

4) Any single hump distribution $\psi_o = \frac{N}{2\pi} F(p_o)\,, \quad \int_{-\infty}^{\infty} F(p_o)\, dp_o = 1$

Stability criteria (eq. E-15, E-16)

$$\frac{1}{\gamma_g^2} < 0 \quad \text{or} \quad \frac{\Delta}{2} > \sqrt{\frac{\nu g \gamma}{\alpha\, \gamma_g^2}}\; m_o c r_o \quad \text{if} \quad \frac{1}{\gamma_g^2} > 0$$

4-5

where $\frac{4}{\Delta^2} = \int \frac{(F_o - F)}{(p_o - p)^2}\, dp$ and $\left.\frac{\partial F}{\partial p}\right)_{p_o} = 0\,,$

$F_o \equiv F(p_o)$

B. <u>TWO-STREAM N.M.I.</u> (eq. 3-13)

1)
$$1 = \frac{e^2 g}{r_o}\left[\int \frac{\frac{\partial \psi_{o+}}{\partial p_o}\, dp_o}{\dot\theta_{o+} - k_+ p_o - \frac{\Omega}{2}} + \frac{1}{\gamma_\sigma^2}\int \frac{\frac{\partial \psi_{o-}}{\partial p_o}\, dp_o}{\dot\theta_{o-} - k_- p_o - \frac{\Omega}{2}}\right]$$

$$+ \left[\frac{1}{\gamma_\sigma^2} - (1+\epsilon)^2\right]\left(\frac{e^2 g}{r_o}\int \frac{\frac{\partial \psi_{o+}}{\partial p_o}\, dp_o}{\dot\theta_{o+} - k_+ p_o - \frac{\Omega}{2}}\right)\left(\frac{e^2 g}{r_o}\int \frac{\frac{\partial \psi_{o-}}{\partial p_o}\, dp_o}{\dot\theta_{o-} - k_- p_o - \frac{\Omega}{2}}\right) \qquad \text{4-6}$$

valid if Im $\Omega > 0$

2) **Pulse function**

$$\psi_{0-} = \psi_{0+} = \begin{cases} \dfrac{N}{2\pi\Delta} & |\rho\theta| \leq \dfrac{\Delta}{2} \\ 0 & |\rho\theta| > \dfrac{\Delta}{2} \end{cases}$$

Dispersion relation (eq. 3-20)

$$\frac{1}{\nu} = \frac{\frac{m_-}{m_+} c^2 \alpha_+ g}{(\Delta v_+)^2 - \left(v_+ - r_0 \frac{\Omega}{2}\right)^2} + \frac{\frac{1}{\gamma_\sigma^2} \frac{c^2 \alpha_- g}{\gamma}}{(\Delta v_-)^2 - \left(v_- - r_0 \frac{\Omega}{2}\right)^2}$$

4-7

$$+ \frac{\frac{\nu}{\gamma} \frac{m_-}{m_+} c^4 \alpha_+ \alpha_- \left[(1+\epsilon)^2 - \frac{1}{\gamma_\sigma^2}\right] g^2}{\left[(\Delta v_+)^2 - \left(v_+ - r_0 \frac{\Omega}{2}\right)^2\right] \cdot \left[(\Delta v_-)^2 - \left(v_- - r_0 \frac{\Omega}{2}\right)^2\right]}$$

3) The above equation may be reduced to the two equations (eq. 3-22)

1. $$\left(r_0 \frac{\Omega}{2} - v_+\right)^2 = (\Delta v_+)^2 - \nu \frac{m_-}{m_+} c^2 \alpha_+ g$$

4-8

Stability criteria (eq. 3-24) $\Delta v_+ > \sqrt{\nu \frac{m_-}{m_+} c^2 \alpha_+ g}$

2. $$\left(r_0 \frac{\Omega}{2} - v_-\right)^2 = (\Delta v_-)^2 - \frac{\nu c^2 \alpha_- g}{\gamma} \left[\frac{1}{\gamma_g^2} - \frac{\nu m_-}{m_+} g \left(\frac{1}{1-h} - 2\right)\right]$$

Stability criteria

$$\frac{1}{\gamma_g^2} - \nu \frac{m_-}{m_+} g \left(\frac{1}{1-h} - 2\right) < 0$$, or if this quantity is positive,

$$\Delta v_- > \sqrt{\frac{\nu c^2 \alpha_- g}{\gamma} \left[\frac{1}{\gamma_g^2} - \frac{\nu m_-}{m_+} g \left(\frac{1}{1-h} - 2\right)\right]}$$

C. THE EFFECT OF BETATRON OSCILLATIONS

The inclusion of the betatron oscillations, non-relativistically shows that the R.H.S. of eq. A and hence ν in eqs. 2, 3, 4 and 5 should be multiplied by

for z oscillations (eq. H-13) $\left\{ 1 - .005\, \eta^2 n \frac{\rho_z^4}{r_o^4} \right.$

4-9

for r oscillations (eq. I-20) $\left\{ \dfrac{1 - \dfrac{\eta^2}{8} \dfrac{1}{1-n} \dfrac{\rho_r^2}{r_o^2}}{1 + \dfrac{3}{2} \dfrac{\nu g c^2}{r_o^2} \dfrac{\eta^2}{(1-n)^2 \dot{\theta}_o^2}} \right.$

if these factors are near unity.

The stability criteria given above are negligibly changed by these factors. All the above quantities are defined on page 123.

Limits of Validity of Results

The results given above are valid when

$$\left| \frac{\Omega}{\eta} - \dot{\theta}_{o\pm} \right| \ll \sqrt{1-n} \left| \frac{\dot{\theta}_{o\pm}}{\eta} \right|$$

and

$$\Delta_+ \ll m_+ v_+ r_o \qquad \rho \ll \frac{r_o}{|\eta|} \quad , \quad \eta \neq 0$$

$$\Delta_- \ll \gamma m_- c r_o \quad ,$$

D. <u>TWO STREAM LONGITUDINAL INSTABILITY</u> (thin beams, eq. 3-35)

1)

$$\frac{1}{1 - \left(\dfrac{\Omega r_o}{\eta c}\right)^2} = \frac{e^2 g}{r_o} \left[\int \frac{\dfrac{\partial \psi_{o+}}{\partial p_o} \, dp_o}{\dot{\theta}_{o+} - \dfrac{\Omega}{\eta} + \dfrac{p_o}{m_+ r_o^2}} + \int \frac{\dfrac{\partial \psi_{o-}}{\partial p_o} \, dp_o}{\dot{\theta}_{o-} - \dfrac{\Omega}{\eta} + \dfrac{p_o}{\gamma^3 m_- r_o^2}} \right]$$

4-10

2) Using a pulse function for the initial velocity distribution

$$\psi_{o\pm} = \begin{cases} \dfrac{N}{2\pi\Delta} & |p_o| \le \dfrac{\Delta}{2} \\ 0 & |p_o| > \dfrac{\Delta}{2} \end{cases}$$

Dispersion relation (eq. 3-36)

$$\frac{1}{\nu} = \frac{\left[1-\left(\frac{\Omega r_0}{l c}\right)^2\right]\frac{m_-}{m_+}c^2 g}{\left(v_+ - r_0\frac{\Omega}{l}\right)^2 - \left(\Delta v_{\parallel +}\right)^2} + \frac{\left[1-\left(\frac{\Omega r_0}{l c}\right)^2\right]\frac{c^2 g}{\gamma^3}}{\left(v_- - r_0\frac{\Omega}{l}\right)^2 - \left(\Delta v_{\parallel -}\right)^2} \qquad \text{4-11}$$

Stability criterion when $v_+ \ll v_-$, $\Delta v_{\parallel +} = \Delta v_{\parallel -} = 0$, $\gamma^3 m_- \gg m_+$

$$\nu g < \frac{m_+}{m_-}\,\frac{1}{4}\left(\frac{\gamma^3 m_-}{m_+}\right)^{1/2}$$

Limits of Validity

$\Delta_- \ll \gamma m_- c r_0$ $\qquad\qquad$ $\rho \ll \frac{r_0}{|l|}$, $l \neq 0$

$\Delta_+ \ll m_+ c r_0$ if $v_+ \ll c$

$\frac{1}{1-n} \ll \frac{1}{\gamma^2}$

The equation is also valid for straight beams, if $\frac{l}{r_0} \to k_z$.
The g factor, a logarithmic term, is then somewhat changed.

E. TWO STREAM LONGITUDINAL INSTABILITY (thick beam, eq. 3-44)

$$k_z^2 = \omega_{p+}^2 \int \frac{\frac{\partial \psi_{0+}}{\partial v}\, dv}{v - \frac{\Omega}{k_z}} + \frac{\omega_{p-}^2}{\gamma^3}\int \frac{\frac{\partial \psi_{0-}}{\partial v}\, dv}{v - \frac{\Omega}{k_z}} \qquad \text{4-12}$$

where v = particle velocity

$$\omega_{p\pm}^2 = \frac{4\pi n_\pm e^2}{m_\pm}$$

and the disturbance has the behavior $\sim e^{ik_z z - i\Omega t}$

Limits of Validity

$\Delta p_+ \ll m_+ c\Gamma$ if $v_+ \ll c$, $\quad \rho \gg \frac{1}{|k_z|}$

$\Delta p_- \ll \gamma m_- c\Gamma$

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APPENDIX I

The Betatron Equations

We here record the relativistic equations for the motion of a charged particle in electric and magnetic fields using the canonical formalism. The Hamiltonian in cylindrical coordinates is:

$$H = c\sqrt{\left(p_r - \frac{e}{c}A_r\right)^2 + \left(\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right)^2 + \left(p_z - \frac{e}{c}A_z\right)^2 + (m_0 c)^2} + e\varphi \qquad \text{A-1.}$$

Using the six Hamiltonian equations of motion, we obtain:

$$\dot{r} = \frac{1}{m}\left(p_r - \frac{e}{c}A_r\right) \qquad \text{A-2a.}$$

$$\dot{\theta} = \frac{1}{mr}\left(\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right) \qquad \text{b.}$$

$$\dot{z} = \frac{1}{m}\left(p_z - \frac{e}{c}A_z\right) \quad , \; m = \gamma m_0 = \frac{H - e\varphi}{c^2} \qquad \text{c.}$$

$$\dot{p}_r = e\frac{\dot{r}}{c}\frac{\partial A_r}{\partial r} + v_\theta \frac{\partial}{\partial r}\left(-\frac{p_\theta}{r} + \frac{e}{c}A_\theta\right) + e\frac{\dot{z}}{c}\frac{\partial A_z}{\partial r} - e\frac{\partial \varphi}{\partial r} \qquad \text{A-3a.}$$

$$\dot{p}_\theta = e\frac{\dot{r}}{c}\frac{\partial A_r}{\partial \theta} + e\frac{v_\theta}{c}\frac{\partial A_\theta}{\partial \theta} + e\frac{\dot{z}}{c}\frac{\partial A_z}{\partial \theta} - e\frac{\partial \varphi}{\partial \theta} \qquad \text{b.}$$

$$\dot{p}_z = e\frac{\dot{r}}{c}\frac{\partial A_r}{\partial z} + \frac{e v_\theta}{c}\frac{\partial A_\theta}{\partial z} + e\frac{\dot{z}}{c}\frac{\partial A_z}{\partial z} - e\frac{\partial \varphi}{\partial z} \qquad \text{c.}$$

If the betatron field is azimuthally symmetric, then it may be represented by $\vec{A}_{ext} = A_\theta \hat{\theta}$, where

$$A_\theta = B_0 r_0 \left(1 + \frac{(1-n)(r-r_0)^2}{2r_0^2} + \frac{n z^2}{2 r_0^2}\right) \qquad \text{A-4.}$$

which, for small $\frac{r-r_0}{r_0}$, $\frac{z}{r_0}$ using $B = \nabla\times A$ gives $B_z = B_0\left(\frac{r_0}{r}\right)^n$. Also equation 4 satisfies $\nabla\times B = \nabla\times(\nabla\times A) = 0$ and obviously also $\nabla\cdot B = \nabla\cdot\nabla\times A = 0$. Equation 4 also gives the familiar 2-1 condition at $r = r_0$.

For the self-fields we have:

$$\varphi(\vec{r},t) = \int \frac{\rho(\vec{r}\,', t - \frac{|r-r'|}{c})}{|r-r'|} dv \; , \quad \vec{A}(\vec{r},t) = \int \frac{\vec{j}(\vec{r}\,', t - \frac{|r-r'|}{c})}{c\,|r-r'|} dv \qquad \text{A-5.}$$

Consider now an equilibrium situation. $A_r, A_z = 0$, and the self-fields A_θ and φ are assumed small enough to be neglected. The equation of motion for the z direction is

$$\dot{p}_z = e \frac{v_\theta}{c} \frac{\partial A_\theta}{\partial z} \qquad \text{A-6.}$$

Equations 2b and 3c become

$$v_\theta = \frac{1}{m}\left(\frac{p_\theta}{r} - \frac{e}{c} A_\theta\right) \qquad \text{A-7.}$$

and

$$p_z = m \dot{z} \quad . \qquad \text{A-8.}$$

p_θ, $r - r_0$ and z are first-order small terms. Thus to lowest order $v_\theta = v_0 = \frac{-e}{mc} B_0 r_0$, a constant.

Inserting these results into equation 6 gives

$$\frac{d}{dt}(m\dot{z}) = -\left(-v_0 \frac{\partial A_\theta}{\partial z}\right) \qquad \text{A-9.}$$

With the same assumptions the equilibrium motion in the r direction becomes, using equations 2a and 3a,

$$p_r = m\dot{r}$$

$$\frac{d}{dt}(m\dot{r}) = v_0 \frac{\partial}{\partial r}\left(-\frac{p_\theta}{r} + \frac{e}{c} A_\theta\right) = -\left(-v_0 \frac{\partial V}{\partial r}\right) \qquad \text{A-10}$$

Note that $V = -\frac{p_\theta}{r} + \frac{e}{c} A_\theta$ plays the role of a potential for the r and z motion, and that v_0 and V are of opposite sign.

APPENDIX II

The Generalized Potential

We show here that the idea of a potential well applies to all magnetic fields with $0 < n < 1$. In particular we will show that

$$\frac{p_\theta}{r} - \frac{e}{c} A_H = \frac{P_s}{r} - \frac{e}{c} A_B \qquad \text{B-1}$$

where A_H is the vector potential due to an axially symmetric magnetic field where $0 < n < 1$ near the orbit but otherwise arbitrary. $P_s \equiv p_\theta - p_{\theta_0}$ and p_{θ_0} correspond to the value of p_θ at the equilibrium orbit $r = r_0$. Thus eq. A-10 describes the radial motion if p_θ is replaced by P_s , and the r and z motion may be described for these arbitrary magnetic fields by the potential $V' = -\frac{P_s}{r} + \frac{e}{c} A_B$.

Proof: We have

$$\vec{B} = \nabla \times \vec{A}, \quad B_z = \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) = \frac{A_\theta}{r} + \frac{\partial A_\theta}{\partial r}, \text{ and } \quad B_z = B_0 \left(\frac{r_0}{r}\right)^n \qquad \text{B-2}$$

Thus

$$A_\theta = \frac{1}{r}\int_0^r B_z r\, dr$$

$$= \frac{1}{r}\int_0^{r_0} B_z r\, dr + \frac{1}{r}\int_{r_0}^r B_z r\, dr$$

and in particular if B_z is given by eq. 2

$$A_H = \frac{\phi_0}{r} + \frac{B_0 r_0^n}{r}\left(\frac{r^{2-n}}{2-n} - \frac{r_0^{2-n}}{2-n}\right). \qquad \text{B-3}$$

Since the expression for B_z in eq. 2 has been used, which is valid only near r_0, A_θ in eq. 3 is also valid only near r_0 . The equilibrium orbit is found from eq. A-3a, with $\dot{p}_r = \ddot{r} = 0$. Since in the equilibrium situation $A_r^{\circ}, A_z^{\circ} = 0$, and $A_\theta^{\circ}, \varphi^{\circ}$ are neglected, the equilibrium orbit is obtained from

$$\frac{\partial}{\partial r}\left(\frac{p_\theta}{r} - \frac{e}{c} A_\theta\right) = 0 \quad ,$$

or

$$- \frac{p_0}{r^2} - \frac{e}{c} \frac{\partial A_\theta}{\partial r} = 0$$

Therefore

$$\frac{p_{0_0}}{r_0^2} = - \frac{e}{c} \left[\frac{\partial A}{\partial r} \right]_{r = r_0} = \frac{e}{c} \frac{\phi_0}{r_0^2} - \frac{e}{c} B_0 \qquad \text{B-4}$$

using also eq. 3.

It is now possible to evaluate $\frac{p_0}{r} - \frac{e}{c} A_H$. The substitution

$$P_s = p_0 - p_{0_0} \qquad \text{B-5}$$

and the expansion

$$\frac{1}{r^n} = \frac{1}{r_0^n} - \frac{n}{r_0^{n+1}} (r - r_0) + \frac{n(n+1)}{r_0^{n+2}} \frac{(r - r_0)^2}{2} \qquad \text{B-6}$$

are needed together with eqs. 3 and 4. Thus

$$\frac{p_0}{r} - \frac{e}{c} A_H = \frac{P_s + p_{0_0}}{r} - \frac{e}{c} \left[\frac{\phi_0}{r} + \frac{B_0 r_0^n}{2-n} \left(r^{1-n} - \frac{r_0^{2-n}}{r} \right) \right]$$

$$= \frac{P_s}{r} - \frac{e}{c} \frac{B_0 r_0^2}{2-n} \left((1-n) \frac{1}{r} + r_0^{n-2} r^{1-n} \right) \qquad \text{B-7}$$

$$= \frac{P_s}{r} - \frac{e}{c} B_0 r_0 \left(1 + \frac{(1-n)(r - r_0)^2}{2 r_0^2} \right) .$$

A_H as given by eq. 3 is incomplete. Since $\nabla \cdot B = 0$ there is also a field component $B_r = - \frac{\partial A_0}{\partial z}$. This term is $- \frac{e}{c} B_0 r_0 \frac{n z^2}{2 r_0^2}$. When this term is added to the R.H.S. of eq. 7, we obtain, by comparison with eq. A-4,

$$\frac{p_0}{r} - \frac{e}{c} A_H = \frac{P_s}{r} - \frac{e}{c} A_B .$$

Q. E. D.

APPENDIX III

Equations for the Self-fields

By eq. A-5

$$\varphi(r',\theta',z',t) = e\int \frac{\psi(r,\theta,z,t-\frac{r-r'}{c}, p_r, p_\theta, p_z)\,dr\,d\theta\,dz\,dp_r\,dp_\theta\,dp_z}{|r-r'|} \quad \text{C-1}$$

We shall show below that when the minor beam radius is much less than the perturbation wavelength, that

$$\varphi^{1n} = \frac{e\,g_1}{r_0}\int \psi^{1n}(r,z,p_r,p_\theta,p_z)\,dr\,dz\,dp_r\,dp_z\,dp_\theta \quad \text{C-2}$$

We write ψ as a fourier transform as indicated in eq. 2-18. For a typical term the right hand side of eq. 1 is

$$I = \int \frac{\psi^{1n}(r,z,p_r,p_\theta,p_z)}{|r-r'|}\, e^{il\theta - i\Omega t + i\frac{\Omega}{c}|r-r'|}\,dr\,dz\,d\theta\cdot dp_r\,dp_z\,dp_\theta \quad \text{C-3}$$

$$= \int A\,\frac{e^{il\theta + i\frac{\Omega}{c}|r-r'|}}{|r-r'|}\,d\theta$$

where A is independent of θ. Consider now the physical situation. There is a thin beam of approximately circular cross section with a charge density which varies slightly over the cross section. There is a sinusoidal variation of charge as θ goes through 2π, which gives a corresponding variation in potential. Since it is $\frac{\partial\varphi}{\partial\theta}$ that we wish, it is possible to make a few approximations. Since the beam is thin, the potential varies little across the beam. Thus the observation point $\vec{r}'$ may be chosen at the center of the beam, i.e., $r' = r_0$, $z' = 0$. Since the potential is approximately constant over the cross section we shall also neglect the transverse motion implied by the constraint equation, eq. 2-6. Thus

$$|r-r'| = \left[r^2 + r_0^2 - 2rr_0\cos(\theta-\theta') + z^2\right]^{1/2} = \frac{2\sqrt{rr_0}}{k}\left[1 - k^2\cos^2\frac{(\theta-\theta')}{2}\right]^{1/2} \quad \text{C-4}$$

where

$$k^2 = \frac{4 r r_0}{(r + r_0)^2 + z^2} \approx 1 - \frac{(r - r_0)^2 + z^2}{4 r_0^2} \quad \text{if} \quad \frac{(r - r_0)^2 + z^2}{4 r_0^2} \ll 1 \ .$$

Since $r \approx r_0$ for a thin beam, the θ integral is a function only of $(r - r_0)^2 + z^2$. The r, z dependence of ψ''' is weak. Thus the surfaces of constant I are cylinders about the center of the beam. Because of this symmetry, the potential at the beam surface is given by evaluating the integral at $(r - r_0)^2 + z^2 = \rho^2$ and assuming that all the charge within the cylinder is located at the line $r = r_0 , z = 0$. This is similar to the approximations made in calculating the inductance of a thin coil. Therefore

$$k^2 = 1 - \frac{\rho^2}{4 r_0^2} \tag{C-5}$$

and eq. 3 may be written as

$$I = \int \frac{A e^{i l \theta} e^{i \frac{\Omega}{c} \cdot 2 r_0 \left[1 - k^2 \cos^2 \left(\frac{\theta - \theta'}{2}\right)\right]^{1/2}}}{2 r_0 \left[1 - k^2 \cos^2 \left(\frac{\theta - \theta'}{2}\right)\right]^{1/2}} d\theta \tag{C-6}$$

We have also set $k \approx 1, r \approx r_0$. Next we make the substitution $\theta - \theta' = \alpha$ which gives

$$I = e^{i l \theta'} A \int_{-\pi}^{\pi} \frac{e^{i l \alpha} e^{i \frac{\Omega}{c} 2 r_0 \left[1 - k^2 \cos^2 \frac{\alpha}{2}\right]^{1/2}}}{2 r_0 \left[1 - k^2 \cos^2 \frac{\alpha}{2}\right]^{1/2}} d\alpha \tag{C-7}$$

Since $e^{i l \alpha} = \cos l\alpha + i \sin l\alpha$, and the rest of the integrand is even in $\cos \alpha$, the $\sin \alpha$ term gives zero, so that

$$I = e^{i l \theta'} A \int_{-\pi}^{\pi} \frac{\cos l\alpha \, e^{i \frac{\Omega}{c} 2 r_0 \left[1 - k^2 \cos^2 \frac{\alpha}{2}\right]^{1/2}}}{2 r_0 \left[1 - k^2 \cos^2 \frac{\alpha}{2}\right]^{1/2}} d\alpha \tag{C-8}$$

Note now that the major contribution to the integral occurs when the denominator is zero. Thus the contribution of the exponent is small. If desired, this term may be evaluated by expanding $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ This will give a negative imaginary contribution to Ω , caused by radiation damping, which is however smaller than the growth rate for the unstable case, as is seen by evaluating the first term in the expansion for the exponent and using eq. G-3.

The remaining integral is now evaluated by elliptic functions. We have for $l = 1$,

$$4L_1 = \int_{-\pi}^{\pi} \frac{\cos\alpha \, d\alpha}{(1 - k^2 \cos^2\frac{\alpha}{2})^{1/2}}$$

$$= 2\int_0^{\pi} \frac{\cos\alpha \, d\alpha}{(1 - k^2 \cos^2\frac{\alpha}{2})^{1/2}} \qquad \text{C-9}$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{[-\cos^2\varphi + \sin^2\varphi]\, d\varphi}{(1 - k^2 \sin^2\varphi)^{1/2}} = 4\int_0^{\frac{\pi}{2}} \frac{(1 - 2\cos^2\varphi)\, d\varphi}{(1 - k^2 \sin^2\varphi)^{1/2}}$$

where we have written $\frac{\alpha}{2} = \frac{\pi}{2} - \theta$. Thus

$$4L_m \equiv \int_{-\pi}^{\pi} \frac{\cos^m\alpha \, d\alpha}{(1 - k^2 \cos^2\frac{\alpha}{2})^{1/2}} = 4\int_0^{\frac{\pi}{2}} \frac{(1 - 2\cos^2\varphi)^m \, d\varphi}{(1 - k^2 \sin^2\varphi)^{1/2}} \qquad \text{C-10}$$

This integral may be evaluated easily, for $m = 0, 1$ using elliptic functions. (25)
For $m = 2, 3 \ldots$ one makes the substitution $z = \sin\varphi$, and using pp. 181-2, (25)
these may also be evaluated in term of the elliptic fct. K, E. Since $k \approx 1$, we set E, k = 1 in these evaluations. K (k) has a logarithmic dependence. After some calculation we obtain

$$L_0 = K \;, \qquad K = \ln \frac{8 r_0}{\rho}$$

$$L_1 = K - 2 \qquad L_3 = K - \frac{34}{3 \cdot 5} \qquad \text{C-11}$$

$$L_2 = K - \frac{4}{3} \qquad L_4 = K - \frac{184}{3 \cdot 5 \cdot 7}$$

The integral in eq. 8, with the neglect of the exponent, is

$$E_l \equiv 2\int_0^{\pi} \frac{\cos l\alpha \, d\alpha}{\left[1 - k^2 \cos^2\frac{\alpha}{2}\right]^{1/2}} \; . \qquad \text{C-12}$$

Since $\cos l\alpha = \sum_{k=0}^{l} a_k \cos^m\alpha$, eq. 12 may be written as the sum of integrals of the form of eq. 10, whose values are given by eq. 11. Thus, after doing the algebra,

$$E_0 = 4K \qquad\qquad E_3 = 4\left(K - \frac{46}{15}\right)$$

$$E_1 = 4(K-2) \qquad\qquad E_4 = 4\left(K - \frac{8.44}{105}\right) \qquad \text{C-13}$$

$$E_2 = 4\left(K - \frac{8}{3}\right) \quad \text{or} \quad E_l \approx 4(K - \ln l - 2 + .04)$$
$$= 4\left(\ln \frac{r_0}{l\rho} + .12\right), \quad l \neq 0 .$$

Thus eq. 8 becomes

$$I_l = e^{il\theta'} \frac{A\, E_l}{2 r_0} . \qquad \text{C-14}$$

Thus through eq. 3, the right hand side of eq. 1 is

$$\text{R.H.S eq. } 1 = \int d\Omega \sum_l \left[\frac{e g_l}{r_0} \psi^{l\Omega} dr\, dz\, dp_r dp_z dp_\theta \right] e^{il\theta - i\Omega t} \qquad \text{C-15}$$

where $g_l = \frac{E_l}{2}$. The left hand side of eq. 1 may also be fourier analyzed, as in eq. 2-18 . Since eq. 1 is true for any l and Ω , the integrands must be equal. Hence eq. 2 is proved and

$$g_l = \frac{E_l}{2} , \qquad K = \ln \frac{8 r_0}{\rho} \qquad \text{C-16}$$

Next we use the above method to evaluate an integral of the form,

$$A(r',\theta',z',t') = e \int \frac{\cos(\theta - \theta')}{|r - r'|} J\, dr\, d\theta\, dz\, d\vec{p} \qquad \text{C-17}$$

By the same procedure, eq. 17 may be written in the form of eq. 7. Since $\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$ in eq. 7, instead of the term $e^{il\alpha}$, we have $\frac{e^{i(l+1)\alpha} + e^{i(l-1)\alpha}}{2}$. Thus fourier analyzing eq. 17, we see that it may be written as

$$A^{l\Omega} = \frac{e\,\hat{g}_l}{r_0} \int J^{l\Omega}(r,z,p_r,p_\theta,p_z)\, dr\, dz\, dp_r\, dp_\theta\, dp_z \qquad \text{C-18}$$

where now

$$\hat{g}_l = \frac{1}{2} \cdot \frac{E_{l-1} + E_{l+1}}{2} . \qquad \text{C-19}$$

Note that $\hat{g}_1 > g_1$ from the concavity of the curve in Fig. 6 .

For completeness, we note that if the r, z, p_r, p_z parts of eq. 2 and 18 are integrated over, we obtain

$$\varphi^{1\Omega} = e\,\frac{g_1}{r_0}\int \psi^{1\Omega}(p_0)\,dp_0 \qquad \text{C-20}$$

$$A^{1\Omega} = e\,\frac{\hat{g}_1}{r_0}\int J^{1\Omega}(p_0)\,dp_0 \quad .$$

APPENDIX IV

Derivation of $\dot\theta = \dot\theta_0 - k p_\theta$

We here expand $\dot\theta$ in powers of p_θ, where

$$\dot\theta = \frac{1}{mr}\left(\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right) \qquad \text{D-1}$$

and A_θ is given by eq. A-4. Since we assume that p_θ is small we keep only the first term in the expansion. Thus we write

$$\dot\theta = \dot\theta_{p_\theta = 0} + \left(\frac{\partial\dot\theta}{\partial p_\theta}\right)_{p_\theta = 0} \cdot p_\theta \qquad \text{D-2}$$

and we wish to evaluate $\frac{\partial\dot\theta}{\partial p_\theta}$. Note that $\dot\theta$ is a function of p_θ explicitly, and also implicitly through the dependence of r on p_θ, because $r = f(p_\theta)$ through the constraint equation $r - r_0 = -\frac{p_\theta}{1-n}\frac{c}{e B_0 r_0}$. Also from the Hamiltonian Equation A-1

$$mc = c\sqrt{\left(\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right)^2 + (m_0 c)^2} \qquad \text{D-3}$$

which gives $m = f(p_\theta)$. Thus

$$\frac{\partial\dot\theta}{\partial p_\theta} = \frac{\partial}{\partial p_\theta}\left(\frac{1}{mr}\right)\left[\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right] + \left(\frac{1}{mr}\right)\frac{\partial}{\partial p_\theta}\left[\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right]$$

$$= \left[\frac{1}{r}\left(-\frac{1}{m^2}\right)\frac{\partial m}{\partial p_\theta} + \frac{1}{m}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial p_\theta}\right]\cdot\left[\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right] \qquad \text{D-4}$$

$$+ \frac{1}{mr}\left[\frac{1}{r} + \cancel{\frac{\partial}{\partial r}\left(\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right)\frac{\partial r}{\partial p_\theta}}\right]$$

Using eq. 3,

$$\frac{\partial m}{\partial p_\theta} = \frac{1}{mc^2}\left(\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right)\left(\cancel{\frac{\partial}{\partial r}\left[\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right]\frac{\partial r}{\partial p_\theta}} + \frac{1}{r}\right)$$

$$= \frac{1}{mc^2 r}\left(\frac{p_\theta}{r} - \frac{e}{c}A_\theta\right). \qquad \text{D-5}$$

The crossed out terms are zero by the constraint equation. Setting now $p_\theta = 0$, which gives $r = r_o$, and using the constraint equation, $\frac{\partial r}{\partial p_\theta} = \frac{-c}{(1-n)eA_o}$, we obtain from eqs. 4 and 5, (setting $r_o \equiv r$)

$$\left(\frac{\partial \dot\theta}{\partial p_\theta}\right)_{p_\theta = 0} = -\frac{1}{m^2 r}\cdot\frac{1}{mc^2 r}\left(\frac{eA_\theta}{c}\right)^2 - \frac{1}{mr^2}\left(\frac{-c}{(1-n)eA_o}\right)\cdot\left(\frac{-eA_\theta}{c}\right) + \frac{1}{mr^2} \qquad \text{D-6}$$

$$= -\frac{1}{mr^2}\left(\frac{eA_\theta}{mc}\right)^2\frac{1}{c^2} - \frac{1}{mr^2(1-n)} + \frac{1}{mr^2}$$

$$= \frac{1}{mr^2}\left(-\frac{1}{1-n} + \frac{1}{\beta^2}\right) \equiv -k\,, \qquad (k > 0)$$

where we have neglected the slight r dependence of A_θ and also written $v_o = -\frac{eA_o}{mc}$. Thus eq. 2 is

$$\dot\theta = \dot\theta_o - k\,p_\theta$$

Q.E.D.

APPENDIX V

Nyquist Diagram

The Nyquist diagram technique as used by Penrose[22] is applied here to the N.M.I. dispersion relations, eq. 2-39, to obtain generalized criteria for stability.

To simplify the algebra we make the following substitutions in eq. 2-39. We write

$$\dot{\theta}_0 - k p_0 = -v \qquad \text{and} \qquad \psi_0 = \frac{N}{2\pi}\frac{f}{K} \tag{E-1}$$

and obtain

$$l = \int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial v}}{v + \frac{\Omega}{l}}\, dv \tag{E-2}$$

where

$$K = \frac{N e^2 q}{2\pi r_0 \gamma_g^2} \tag{E-3}$$

Note that $\int_{-\infty}^{\infty} f\, dv = 1$. We assume also that $\gamma_g^2 > 0$.

We will now find the condition that Ω has no positive imaginary part, as a positive imaginary part means that any perburbation grows exponentially.

The quantity l may be a positive or negative integer. Suppose that $l = -|l|$. Then eq. 3 is

$$-l = \int \frac{\frac{\partial f}{\partial v}}{v - \frac{\Omega}{|l|}}\, dv \quad . \tag{E-4}$$

For fixed ℓ , the right hand side of the equation defines a function of Ω , for Ω with positive imaginary part. Let us call this function Z_- . Thus

$$Z_-\left(\frac{\Omega}{|\ell|}\right) = \int \frac{\frac{\partial f}{\partial v}\, dv}{v - \frac{\Omega}{|\ell|}} \quad . \qquad \text{E-5}$$

Consider now the imaginary Ω plane, upon which is drawn the curve C. See fig. 7 . As the dotted portion of the curve goes to infinity, it encloses the positive imaginary plane. By eq. 5, this curve may be mapped into a curve in the W plane. Since the function Z_- is analytic in the upper half plane with no poles, the curve C is mapped into a curve D-, which is also described counterclockwise and encloses all positive imaginary values of Ω , some of which may be enclosed more than once by counterclockwise loops.

On the dotted portion of the curve C, $Z_- = 0$. Thus to obtain an explicit representation of D-, we must obtain the value of Z_- for Ω with a vanishing imaginary part. By, e.g., J. D. Jackson (21) this is

$$Z_-\left(\frac{\Omega}{|\ell|} + i0\right) = P \int \frac{\frac{\partial f}{\partial v}\, dv}{v - \frac{\Omega}{|\ell|}} + i\pi \frac{\partial f}{\partial v}\left(\frac{\Omega}{|\ell|}\right) \qquad \text{E-6}$$

The curve D- described by eq. 6, is sketched in Fig. 8 for a Maxwellian distribution. The curve is labelled with values of real Ω . This curve is from O. Penrose. (22) The interior of the curve encloses values of positive imaginary Ω . For distributions other than Maxwellian the curve is asymmetrical, and may have additional loops, so that positive imaginary values of Ω may be enclosed more than once.

If η in eq. 3 is positive, then the right side of eq. 3 defines a function Z_+,

$$Z_+\left(\frac{\Omega}{\eta}\right) = \int \frac{\frac{\partial f}{\partial v}}{v + \frac{\Omega}{\eta}}\, dv \qquad \text{E-7}$$

The curve C is mapped onto the W plane by the function Z_+. Again the dotted portion of curve C corresponds to $Z_+ = 0$, so that we must find the value of Z_+ for vanishing imaginary part of Ω, in order to find the explicit expression for D+ on the plane W. Thus

$$Z_+\left(\frac{\Omega}{\eta} + i0\right) = P\int \frac{\frac{\partial f}{\partial v}\, dv}{v + \frac{\Omega}{\eta}} - i\pi \frac{\partial f}{\partial v}\left(-\frac{\Omega}{\eta}\right) \qquad \text{E-8}$$

The curve D+ on the W plane is also counterclockwise and encloses all the values of positive imaginary Ω, enclosed by the curve C in the Ω plane. For a Maxwellian distribution, or any symmetrical distribution of v, the curves D- and D+ are identical. For unsymmetrical distributions the curves are related as follows: If we replace Ω by $-\Omega$ in eq. 6, we still get D- but traversed in the opposite direction. The function Z_- is now like Z_+ except that the imaginary parts have opposite sign. Thus if we now reflect the curve D- about the imaginary axis, we will get D+, traversed in the correct counterclockwise rotation.

We can now establish the generalized stability criterion. The dispersion relation, eq. 3 is $Z = -1$. Thus if the point -1 lies inside the curve D, then we have instability. Thus for stability

$$Z_0 > -1 \qquad \text{E-9}$$

where z_0 is the left-most point on the boundary of D on the negative real axis, (see Fig. 8), because then the point $z = -1$ will occur outside the curve D. (In Fig. 9 this left-most point can be either point 1 or 2 and this case will be covered below.) At this point z_0 is real and negative, hence the imaginary term in eq. 6 and 8 is zero. Thus

$$\left(\frac{\partial f}{\partial v}\right)_{v_0 = -\frac{\Omega}{2}} = 0 \qquad \text{for } \pm 2 \qquad \text{E-10}$$

Since the imaginary part of z goes from positive to negative as we traverse the point z_0 in the counter-clockwise direction on the curve D, that is increasing values of Ω , the solution of eq. 10 corresponds to a maximum in f.

Inserting the solution of eq. 10, $v_0 = -\frac{\Omega}{2}$ into eq. 6 or 8, now gives for the negative real part of z_0 ,

$$z_0 = P\int \frac{\frac{\partial f}{\partial v}}{v - v_0}\, dv \qquad \text{E-11}$$

Since eq. 10 is true, eq. 11 may be transformed into an integral without the principal part, (see O. Penrose[(22)])

$$z_0 = P\int \frac{\frac{\partial f}{\partial v}}{v - v_0}\, dv = \int \frac{f - f_0}{(v - v_0)^2}\, dv \qquad \text{E-12}$$

This last expression is finite because $f = f_0 + f_0''(v - v_0)^2 + \cdots$. Inserting eq. 11 into 9 we obtain finally

$$\frac{1}{\int \frac{f_0 - f}{(v_0 - v)^2}\, dv} > 1 \quad , \quad \left(\frac{\partial f}{\partial v}\right)_{v = v_0} = 0 \quad , \quad \left(\frac{\partial^2 f}{\partial v^2}\right)_{v = v_0} < 0 \qquad \text{E-13}$$

as the condition for stability, for a single peaked distribution. The left side of the above inequality is positive, and implies that if the width of the velocity distribution is large enough the beam is stable. (If $\gamma_g^2 < 0$ in eq. 3, then the 1 in the inequality is replaced by -1 and the inequality is always true, so even narrow beams are stable).

This condition is necessary and sufficient only for single peaked distributions, such as in Fig. 8 . For other distributions, as in Fig.9 , it is possible to locate all the points on the axis and by determining whether they correspond to maximum or minimum, one can determine the sense of the curve D running through them. This is sufficient to determine whether or not the region between any two points lies inside or outside the curve. One simply draws any counter-clockwise curve connecting the points in any desired order, but such that the sense of the curve is correct. We have not found any theorem proving this, but the reader may easily draw any number of figures to convince himself.

We now rewrite eq. 13. Using the definition of f , making the transformation to p_0 again, we now redefine a new $F = \frac{f}{K} = \frac{2\pi}{N}\psi_0$ normalized to 1 on the field p_0 . Thus

$$\frac{1}{k\int\frac{F_0 - F}{(kp_0 - kp)^2}\,dp} > K$$

or

$$\frac{1}{\frac{\gamma m_0 r^2}{\alpha}\int\frac{F_0 - F}{(p_0 - p)^2}\,dp} > \frac{v q m_0 c^2}{\gamma_g^2} . \qquad \text{E-14}$$

We define

$$\int \frac{F_0 - F}{(p_0 - p)^2} dp \equiv \frac{4}{(\Delta p)^2} \quad . \tag{E-15}$$

Then we can write eq. 14 as

$$(\Delta p)^2 > \frac{4 \nu q r}{r_g^2 \alpha} (m_0 c r)^2 \tag{E-16}$$

which gives the criterion for stability.

Penrose has a plot of $Z\left(\frac{\Omega}{?}\right)$, when $F\left(= \frac{f}{K} = \frac{2\pi\psi}{N}\right)$ is Maxwellian, and

$$F = \frac{1}{\alpha \sqrt{\pi}} e^{-\frac{p^2}{\alpha^2}} \tag{E-17}$$

The negative of the left hand side of eq. 15 is then the left-most point of this plot. To obtain the correct units, one sets $w^2 = 1$, in Penrose's figure, with the result

$$-(\text{L.H.S. Eq. 15}) = -\frac{2}{\alpha^2}$$

or

$$\Delta p = \sqrt{2}\,\alpha \tag{E-18}$$

using the right hand side of eq. 15. This result inserted into eq. 16 gives the stability criterion for Maxwellian distributions.

APPENDIX VI

Resonance Function

We solve eqs. 2-39 for the resonance distribution function,

$$\psi_0 = \frac{N}{2\pi} \frac{\Delta}{\pi} \frac{1}{p_0^2 + \Delta^2} \tag{F-1}$$

This function is normalized to N, the total number of particles on the field θ, p_0. To simplify the algebra we use eqs. E-1 and E-3 and define $f = FK$, so that $\psi_0 = \frac{N}{2\pi} F$. Then we obtain

$$F = \frac{\Delta}{\pi} \frac{1}{\left(\frac{\dot{\theta}_0 + v}{k}\right)^2 + \Delta^2} = \frac{k^2 \Delta}{\pi} \frac{1}{(\dot{\theta}_0 + v)^2 + (k\Delta)^2} . \tag{F-2}$$

Next we write $F = kf'$, $k\Delta = \Delta'$ so that eq. 2-39a becomes

$$1 = -Kk \int_{-\infty}^{\infty} \frac{\frac{\partial f'}{\partial v} dv}{v + \frac{\Omega}{l}} \tag{F-3}$$

or

$$1 = -Kk(-2) \int_{-\infty}^{\infty} \frac{(\dot{\theta}_0 + v) \; dv}{\left(v + \frac{\Omega}{l}\right)\left[(\dot{\theta}_0 + v)^2 + \Delta'^2\right]} \tag{F-4}$$

We shall evaluate the integral using the residue theorem. This equation is valid only if Ω has a positive imaginary part. Thus the integral is evaluated differently depending on whether l is a positive or negative number. Let us first assume that $l > 0$. Then the poles of the integrand in the upper v plane occur only at $v_1 = -\dot{\theta}_0 + i\Delta'$ which is a pole of order 2. We will evaluate the integral now by closing the contour in the upper v plane. See Fig. 10. (Closing the contour in the lower half-plane is also permissible, but because there is now a pole due to $-\frac{\Omega}{l}$, the evaluation of the residue is more cumbersome.) Then

$$\int_{-\infty}^{\infty} \frac{(\dot{\theta}_0 + v)\, dv}{\left(v + \frac{\Omega}{2}\right)\left[(\dot{\theta}_0 + v)^2 + \Delta'^2\right]} = 2\pi i \sum \text{Res (upper half-plane)} \qquad \text{F-5}$$

where

$$\text{Res}\Big|_{v=v_1} = \frac{d}{dv}(v - v_1)^2 \left[\frac{\dot{\theta}_0 + v}{\left(v + \frac{\Omega}{2}\right)(\dot{\theta}_0 + v + i\Delta')^2(\dot{\theta}_0 + v - i\Delta')^2}\right]_{v=v_1}$$

$$= \frac{d}{dv}\, \frac{\dot{\theta}_0 + v}{\left(v + \frac{\Omega}{2}\right)(\dot{\theta}_0 + v + i\Delta')^2} \qquad \text{F-6}$$

$$= \left[\frac{(\dot{\theta}_0 + v + i\Delta')^2\left(v + \frac{\Omega}{2}\right) - (\dot{\theta}_0 + v)\left[(\dot{\theta}_0 + v + i\Delta')^2 + \left(v + \frac{\Omega}{2}\right) \cdot 2 \cdot (\dot{\theta}_0 + v + i\Delta')\right]}{\left(v + \frac{\Omega}{2}\right)^2 (\dot{\theta}_0 + v + i\Delta')^4}\right]_{v=v_1}$$

$$= \frac{-i\Delta'(2i\Delta')^2}{\left(v + \frac{\Omega}{2}\right)^2 (2i\Delta')^4} = \frac{i\Delta'}{\left(-\dot{\theta}_0 + i\Delta' + \frac{\Omega}{2}\right)^2 4\Delta'^2}$$

Inserting this result into eq. 5 and 6 gives

$$1 = \frac{2kK\Delta'}{\pi}\, \frac{-2\pi\Delta'}{\left(-\dot{\theta}_0 + i\Delta' + \frac{\Omega}{2}\right)^2 4\Delta'^2}$$

or

$$\left(-\dot{\theta}_0 + i\Delta' + \frac{\Omega}{2}\right)^2 + kK = 0 \quad .$$

Separating this equation into real and imaginary parts and setting $A = \frac{\Omega_r}{2} - \dot{\theta}_0$, $B = \frac{\Omega_i}{2} + \Delta'$ gives the two equations,

$$A^2 - B^2 + kK = 0 \qquad \text{F-7}$$

$$2iAB = 0 \qquad \text{F-8}$$

Eq. 8 may be satisfied by setting A or B equal to zero. Since $\Delta' > 0$, and we have assumed $\Omega_i > 0$, B cannot equal zero. Thus the solutions to the two equations are $A = 0$, $B = \pm\sqrt{kK}$ or $\frac{\Omega_i}{2} = -\Delta' \pm \sqrt{Kk}$. Again since $\Omega_i > 0$, the solution is $\frac{\Omega_i}{2} = \sqrt{Kk} - \Delta'$, valid only if $\sqrt{Kk} > \Delta'$. Thus the complete solution is

$$\frac{\Omega}{2} = \dot{\theta}_0 + i\left(\sqrt{Kk} - \Delta'\right) \quad . \qquad \text{F-9}$$

If $l < 0$, then the pole at $-\frac{\Omega}{l}$ is in the upper half-plane and we close the contour in the lower half-plane. The only pole in this region is at $v_2 = -\dot{\theta}_0 - i\Delta'$. The integral in eq. 5 now equals the negative of the residue in the lower plane because the contour is described clockwise. One sees easily that now

$$\mathrm{Res}\Big|_{v = v_2} = \frac{-i\Delta'}{\left(-\dot{\theta}_0 - i\Delta' + \frac{\Omega}{l}\right)^2 \, 4\Delta'^2} \quad . \tag{F-10}$$

Since the integral in eq. 5 is minus this and proceeding as above we obtain again the eqs. 7 and 8 where $A' = \frac{\Omega_r}{-|l|} - \dot{\theta}_0$, $B' = \frac{\Omega_i}{-|l|} - \Delta'$. Again B' cannot equal zero because $\Omega_i > 0$ is assumed and hence $A' = 0$ and $B' = \pm\sqrt{kK}$, or $\frac{\Omega_i}{-|l|} = \Delta' \pm \sqrt{Kk}$. Again since $\Omega_i > 0$, $\frac{\Omega_i}{-|l|} = \Delta' - \sqrt{Kk}$ valid only if $\sqrt{Kk} > \Delta'$.

Combining this result with that of eq. 9, we have

$$\frac{\Omega}{l} = \dot{\theta}_0 \pm i\left(\sqrt{Kk} - \Delta'\right) \quad , \quad \sqrt{Kk} > \Delta' \tag{F-11}$$

and the plus sign is valid if $l > 0$, and the minus sign for $l < 0$.

Next we solve the dispersion relation for the damped solutions. There is now an additional term which must be added to the right side of eq. 4 as is evident from eq. 2-39b. Using the transformations, $\psi_0 = \frac{N}{2\pi} k f'$, eq. E-1 and E-3 and $k\Delta = \Delta'$, this term is

$$2\pi i (Kk) \frac{\partial f'}{\partial v}\left(-\frac{\Omega}{l}\right) = 2\pi i (Kk)(-2) \frac{\Delta'}{\pi} \frac{\dot{\theta}_0 + v}{\left[(\dot{\theta}_0 + v)^2 + \Delta'^2\right]} \Bigg|_{v = -\frac{\Omega}{l}} \quad . \tag{F-12}$$

We assume that $l > 0$ and perform the integration as in eq. 5. Since $\frac{\Omega}{l}$ now has a negative imaginary part, there is a pole at $v = v_3 \equiv -\frac{\Omega}{l}$, in the upper half-plane besides the pole at v_1. Thus we have a term

$$2\pi i \operatorname{Res}\Big|_{v=v_3} = \frac{\dot{\theta}_0 + v}{\left[(\dot{\theta}_0 + v)^2 + \Delta'^2\right]^2}\Bigg|_{v=-\frac{\Omega}{l}} \qquad \text{F-13}$$

additional to the Residue at v_1, given in eq. 6. Inserting this term into eq. 4 we see that it cancels exactly the term in eq. 12. The dispersion relation is now again the same as eq. 7 and 8, with solution $A = 0$. B cannot equal zero in eq. 8 because then A is imaginary from eq. 7. Thus from eq. 7, $B = \pm\sqrt{kK}$ or $\frac{\Omega_i}{l} = -\Delta' \pm \sqrt{Kk}$. Since Ω_i is assumed negative the solutions are

$$\frac{\Omega_i}{l} = -\Delta' - \sqrt{Kk}, \qquad \frac{\Omega_i}{l} = -\Delta' + \sqrt{Kk} \qquad \text{F-14}$$

where the first solution is valid for all values of $\sqrt{Kk}$.

Next we examine the solution for $l = -|l|$. Since the pole due to $-\frac{\Omega}{l}$ is now in the lower half-plane we close the contour in the lower half-plane. The integral eq. 5 is now equal to the negative of the sum of the residues. Thus we have a term equal to the negative of eq. 14 due to the pole at $-\frac{\Omega}{l}$ which again cancels the added term eq. 12, because that term has a minus sign when l is negative (see comment following Eq. 2-39). The other Residue at $v = v_2$ is given by eq. 10. Inserting this result into eq. 4 now gives the eqs. 7 and 8 where A', B' are defined as above. Again, the solution A' = 0 to eq. 8, gives real solutions to B, while the solution B' = 0 does not. From eq. 7 we see that $B' = \pm\sqrt{Kk}$ or $\frac{\Omega_i}{|l|} = \Delta' \pm \sqrt{Kk}$ Since Ω_i is negative we see that the negative sign for the radical is valid only if $\sqrt{Kk} < A'$. Summarizing our results, the roots of eq. 2-39 with a resonance function are

$$\frac{\Omega}{l} = \dot{\theta}_o + i\left(-A' \pm \sqrt{Kk}\right) \qquad \text{F-15}$$

for positive l . For negative l , the i becomes $-i$.

When K is negative, which will occur if $\delta_j^2 < 0$ in eq. 3, eq. 15 is still valid although the derivation given above must be modified. In particular, the quantities $B, B' = 0$ now and $A, A' \neq 0$.

APPENDIX VII

Pulse Function

Here we evaluate the dispersion relation eq. 2-39, using a pulse function distribution for ψ_0, i.e.,

$$\psi_0 = \begin{cases} \dfrac{N}{2\pi}\,\dfrac{1}{\Delta} & \text{for } -\dfrac{\Delta}{2} \le p_\theta \le \dfrac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases} \qquad \text{G-1}$$

Eq. 2-39 now becomes

$$1 = \frac{e^2 q}{\gamma_g^2 r_0}\,\frac{N}{2\pi\Delta}\int \frac{\left[\delta\left(p_\theta + \frac{\Delta}{2}\right) - \delta\left(p_\theta - \frac{\Delta}{2}\right)\right]}{\left(\dot{\theta}_0 - \frac{\Omega}{l}\right) - k\,p_\theta}\,dp_\theta$$

$$= \frac{e^2 q}{\gamma_g^2 r_0}\,\frac{N}{2\pi\Delta}\left[\frac{1}{\left(\dot{\theta}_0 - \frac{\Omega}{l}\right) + k\frac{\Delta}{2}} - \frac{1}{\left(\dot{\theta}_0 - \frac{\Omega}{l}\right) - k\frac{\Delta}{2}}\right]$$

$$= \frac{e^2 q}{\gamma_g^2 r_0}\,\frac{N}{2\pi\Delta}\;\frac{-k\Delta}{\left(\dot{\theta}_0 - \frac{\Omega}{l}\right)^2 - \left(\frac{k\Delta}{2}\right)^2}$$

or

$$\left(\dot{\theta}_0 - \frac{\Omega}{l}\right)^2 = \left(\frac{k\Delta}{2}\right)^2 - \left(\frac{k\,e^2 q\,N}{2\pi r_0\,\gamma_g^2}\right) \qquad \text{G-2}$$

Writing now $k = \dfrac{\alpha}{\gamma m_0 r^2}$, and $\dfrac{N}{2\pi r}\,\dfrac{e^2}{m_0 c^2} = \nu$, eq. 2 becomes

$$\dot{\theta}_0 - \frac{\Omega}{l} = \frac{\pm\alpha}{\gamma m_0 r^2}\left[\frac{\Delta^2}{4} - \frac{\nu q \gamma\,(m_0 c r)^2}{\gamma_g^2\,\alpha}\right]^{1/2} \qquad \text{G-3}$$

Note now that the second of eq. 2-39 also gives the same result. This is because $\dfrac{\partial\psi_0}{\partial p_\theta} \neq 0$ only when $\dfrac{\dot{\theta}_0 - \frac{\Omega}{l}}{k} = \pm\dfrac{\Delta}{2}$, which can occur only if $N = 0$ by eq. 3. Thus eq. 3 is valid for $Im\,\Omega$ positive or negative.

APPENDIX VIII

Betatron Z Oscillations

In this section the method of characteristics is used to solve eq. 2-54. The dispersion relation for the N.M.I., with the inclusion of the axial (z) betatron oscillations is thereby obtained.

Eq. 2-54 is a linear partial differential equation in the variables z, p_z. The standard technique for solving this equation is given in, e.g., Cohen's "Differential Equations."[26] The solution is obtained by solving

$$\frac{dp_z}{-a_2 z} = \frac{dz}{a_3 p_z} = \frac{d\psi^{1n}}{a_1 \varphi^{1n} \frac{\partial \psi_0}{\partial p_0} - (a_0 + b z^2)\psi^{1n}} . \qquad \text{H-1}$$

The solution of the equation obtained from the first equality is $a_3 p_z^2 + a_2 z^2 = c_1$. To find the other solution we use the last equality. This contains p_z. Using c_1, to eliminate p_z, and eq. 2-57, we find that $\psi_0 = A f(c_1)$ and hence is not a function of z any more. Thus from eq. 1 we must now solve, after rearranging:

$$\frac{d\psi^{1n}}{dz} + \frac{(a_0 + b z^2)\psi^{1n}}{\sqrt{a_3(c_1 - a_2 z^2)}} = \frac{a_1 \varphi^{1n} \frac{\partial \psi_0}{\partial p_0}}{\sqrt{a_3(c_1 - a_2 z^2)}} . \qquad \text{H-2}$$

This is a linear differential equation for ψ^{1n} as a function of z. Again using standard techniques as, e.g., in the book by Cohen,[26] we find for the particular solution:

$$\psi^{1n} = \left(a_1 \varphi^{1n} \frac{\partial \psi_0}{\partial p_0}\right) e^{-\int \frac{a_0 + b z^2}{\sqrt{a_3(c_1 - a_2 z^2)}} dz} \left[\int \frac{e^{\int \frac{a_0 + b z^2}{\sqrt{a_3(c_1 - a_2 z^2)}} dz}}{\sqrt{a_3(c_1 - a_2 z^2)}} dz \right] . \qquad \text{H-3}$$

We do not consider the solution of the homogenous equation for which $\phi = 0$ as it does not contribute to the N.M.I. (Note that the first factor is independent of z .)

(27)

We next integrate the exponent. Since $a_3, a_2 > 0$ and using Pierce #121, 132 we obtain:

$$E \equiv \int \frac{a_0 + b z^2}{\sqrt{a_3 (c_1 - a_2 z^2)}} \, dz = \frac{\left(a_0 + \frac{b c_1}{2 a_2}\right)}{\sqrt{a_2 a_3}} \sin^{-1} \sqrt{\frac{a_2}{c_1}} \, z - \frac{b z \sqrt{a_3 (c_1 - a_2 z^2)}}{2 a_2 a_3} \qquad \text{H-4}$$

We can now see that the solution, eq. 3, gives the right limit as the z, ρ_z terms approach zero. From eq. 2-55, $a_2 a_3 = n \dot{\theta}_0^2$. From the definition of c_1 , max $c_1 \approx a_2 \rho^2$. Thus $\frac{b c_1}{a_2} \approx b \rho^2 \approx 2 n \dot{\theta} \frac{\rho^2}{r^2}$ and the second term in E is of order $\frac{\rho^2}{r^2}$. Similarly max $\sqrt{a_3 \rho_z^2} = \sqrt{a_2 \rho^2}$ and the third term in E is also of order $\frac{\rho^2}{r^2}$. Thus the factors of b are of order $\frac{\rho^2}{r^2}$. To obtain the limit of ψ^{in} as $\rho \to 0$, we may therefore let $b \to 0$ in eq. 3. The integration is easily performed to give:

$$\psi^{in} = a_1 \varphi^{in} \frac{\partial \psi_0}{\partial \rho_0} \cdot \frac{1}{a_0} \qquad \text{H-5}$$

which is the solution we would obtain from eq. 2-54 setting all the z terms equal to zero. Thus we recover the correct limit from eq. 3. Any arbitrary constants which appear in the evaluation of eq. 3, may therefore be resolved by noting that eq. 3 must give eq. 5 when $z \to 0$. (ρ^2 is the minor beam radius, while z is a coordinate in the beam.)

We continue to evaluate eq. 3 and perform the integration of the term in the bracket. This may be done easily, only if we assume the exponent E to be small, for then we can expand and write $e^E \approx 1 + E$. By eq. 4 since $\frac{\rho^2}{r^2} \ll 1$, we therefore require also $a_0 \ll \sqrt{a_2 a_3} = \sqrt{n} \, \dot{\theta}_0$. As we shall see below, this implies that in unstable situations the growth rate should be much less than the axial betatron oscillation frequency. This is well satisfied for current densities of interest. Thus writing $e^{E} \approx 1 + E + \frac{E^2}{2}$ we obtain from eq. 4:

$$\int \frac{1 + E + \frac{E^2}{2}}{\sqrt{a_3(c_1 - a_2 z^2)}}\, dz = \frac{1}{\sqrt{a_2 a_3}} \sin^{-1}\sqrt{\frac{a_2}{c_1}}\, z \qquad (1)$$

$$+ \frac{\left(a_0 + \frac{b c_1}{2 a_2}\right)\left[\sin^{-1}\sqrt{\frac{a_2}{c_1}}\, z\right]^2}{2 a_2 a_3} - \frac{b z^2}{4 a_2 a_3} \qquad (E) \qquad \text{H-6}$$

$$+ h_{odd}(z) \qquad \left(\frac{E^2}{2}\right)$$

where $h_{odd}(z)$ is an odd function of z. This eq. 6 differs from the bracket term in eq. 3 by a constant because $\int e^x dx \not\approx \int (1+x) dx$ and differs by a constant, because although $e^x \approx 1 + x$ two indefinite integrals differ by a constant. To find this constant, we let $z \to 0$ in the bracket term of eq. 3, which is equivalent to letting $b \to 0$. Integrating gives

$$[3] = \frac{1}{a_0} e^{\int \frac{a_0}{\sqrt{}} dz} \approx \frac{1}{a_0}\left[1 + \int \frac{a_0}{\sqrt{}} dz + \frac{1}{2}\left(\int \frac{a_0}{\sqrt{}} dz\right)^2\right]$$

These other integrals are trivial and it is seen that eq. 6 differs from $[3]$ by the constant $\frac{1}{a_0}$. We can now evaluate the z dependence of ψ^{in} by expansion of the exponents in eq. 3 which we denote $f^{in}_{(z)}$. Thus

$$f^{in}_{(z)} \approx \left(1 - E + \frac{E^2}{2}\right)\left(\frac{1}{a_0} + \int \frac{\left(1 + E + \frac{E^2}{2}\right)}{\sqrt{}} dz\right)$$

$$\approx \left(1 - E + \frac{E^2}{2}\right)\cdot\frac{1}{a_0} + (1 - E)\int \frac{1}{\sqrt{}} dz + 1\cdot\int \frac{E}{\sqrt{}} dz$$

$$= \underset{1\cdot a_0^{-1}}{\frac{1}{a_0}} + \Big[\underset{1\cdot\int 1}{1} - \underset{-E\cdot a_0^{-1}}{\left(1 + \frac{b c_1}{2 a_0 a_2}\right)}\Big]\frac{\sin^{-1}}{\sqrt{a_2 a_3}} + \underset{-E\cdot a_0^{-1}}{\frac{b}{a_0}\frac{z\sqrt{}}{2 a_2 a_3}} \qquad \text{H-7}$$

$$+ \left[\underset{1\cdot\int E}{\left(a_0 + \frac{b c_1}{2 a_2}\right)} - \underset{-E\cdot\int 1}{2\left(a_0 + \frac{b c_1}{2 a_2}\right)} + \underset{E^2\cdot a_0^{-1}}{\left(a_0 + \frac{b c_1}{2 a_2}\right)^2\cdot\frac{1}{a_0}}\right]\frac{\left[\sin^{-1}\right]^2}{2 a_2 a_3}$$

$$+ \left[\underset{E^2\cdot a_0^{-1}}{\left(a_0 + \frac{b c_1}{2 a_2}\right)\cdot\frac{1}{a_0}} + \underset{-E\cdot\int 1}{1}\right]\frac{b z\sqrt{}\,\sin^{-1}}{2 a_2 a_3\sqrt{a_2 a_3}} + \underset{E^2\cdot a_0^{-1}}{\frac{1}{2 a_0}\left(\frac{b z}{2 a_2 a_3}\right)^2\left(\sqrt{}\right)^2} - \underset{1\cdot\int E}{\frac{b z^2}{4 a_2 a_3}}.$$

Note that when $z \to 0$, $f^{1n} \to \frac{1}{a_0}$, so that eq. 5 is obtained, which indicates that we have correctly chosen the constant. Note too, that therefore $\sin^{-1} z \to 0$ as $z \to 0$, which fixes the branch of this function.

We have kept only those powers of E which give terms $\sim a_0$. Terms which give higher powers of a_0 have been dropped. Eq. 7 contains terms proportional to $\frac{1}{a_0}, \frac{\rho^2}{a_0}, \frac{\rho^4}{a_0}, \rho^2$. The terms proportional to 1 and a_0 only, cancel out. If we keep higher powers of E, the terms in a_0^2, a_0^3, etc., would also cancel because as $\rho^2 \to 0$, $f^{1n} \to \frac{1}{a_0}$ while $\sin^{-1}$ remains finite as $\rho \to 0$. We have also denoted the origin of each term in small script beneath it. Note that f^{1n} is also a function of p_z through $c_1 = a_2 z^2 + a_1 p_z^2$ This c_1 was a constant only in solving the right hand pair of equations in eq. 1, but becomes a variable again, in the solution for ψ^{1n}. This follows from the theory of solving eq. 1.

In eq. 2, we have used the positive square root in writing $a_1 p_z = \sqrt{}$ $= \sqrt{a_1(c_1 - a_2 z^2)}$ Thus f^{1n} in eq. 7 is defined only for positive p_z. Using the negative square root for p_z, we find that $\sin^{-1}$ also changes sign (cf. eq. 4).

*To obtain the dispersion relation we must now insert $\psi^{1n} = a_1 \varphi^{1n} \frac{\partial \psi_0}{\partial p_0} f_{1n}$ into eq. C-2 and φ and ψ represent perturbation quantities if $l \neq 0$. Since we are considering only axial betatron oscillations, but assume the constraint equation linking r and p_0 to be still valid, the integral is only over $dp_0\, dz\, dp_z$. Thus cancelling φ^{1n}, the dispersion relation becomes:

$$1 = \frac{e q a_1}{r_0} \int \frac{\partial \psi_0}{\partial p_0} f^{1n} \, dp_0 \, dz \, dp_z \qquad \text{H-8}$$

ψ_0 is a function of c_1, and hence is even in z, p_z. $\sin^{-1}$ is odd in z, while $\sqrt{} \equiv a_1 p_z$ is odd in p_z. Thus the second and third terms in eq. 7 give zero when integrated. The term in $z\sqrt{}\sin^{-1}$ $= z\,p_z \sin^{-1} z$ is even in z and apparently odd in p_z. As noted above, however, $\sin^{-1}$ changes sign when p_z changes sign and hence $z\sqrt{}\sin^{-1}$ $= z\,p_z \sin^{-1}$ is even. Also because ψ_0 is zero for large p_0 all terms in $f_{1\Omega}$ without the factor a_0 give zero in the p_0 integration. Thus the terms in eq. 7 that contribute to the dispersion relation are:

$$f'_{1\Omega} = \frac{1}{a_0}\left[1 + \left(\frac{b c_1}{2 a_2}\right)^2 \frac{(\sin^{-1})^2}{2 a_2 a_3} + \frac{(b z)^2}{2}\left(\frac{a_3 p_z}{2 a_2 a_3}\right)^2 - \frac{b^2 c_1 z \, a_3 p_z \sin^{-1}}{2 a_2 \cdot 2 a_2 a_3 \sqrt{a_2 a_3}}\right] \qquad \text{H-9}$$

The smallest term in this expression is $\sim \dfrac{\rho^4}{a_0}$. If we compute $f_{1\Omega}$ in eq. 7 to one more power of E, we would get terms $\sim a_0 \rho^2, \rho^4, \dfrac{\rho^6}{a_0}$. Since we want the lowest order non-vanishing term in ρ^2, it seems that we should also keep $a_0 \rho^2$ terms. However, all these terms are odd in z and hence give a zero contribution to the dispersion equation.

We now integrate eq. 8, doing the $dz\,dp_z$ integration first. The z, p_z dependence of ψ_0 is contained in a factor ψ_{0z}, which is independent of p_0, and as shown in eq. 2-57 is a function only of c_1. Suppose for simplicity we define

$$\psi_{0z} = \begin{cases} N & \text{for} \quad c_1 = a_2 z^2 + a_3 p_z^2 \le a_2 \rho^2 \\ 0 & \phantom{\text{for}} \quad c_1 > a_2 \rho^2 \end{cases} \qquad \text{H-10}$$

ψ_{0z} is normalized to unity on the field z, p_z. N is a normalization constant. Thus

$$N \int_A dz\, dp_z = 1 \quad .$$

Since the area of the ellipse $c_1 = a_2 \rho^2$ is $A = \pi \sqrt{\frac{a_2}{a_3}} \rho^2$, we find $N = \frac{1}{A}$. The integrations are now performed most easily by changing to polar coordinates. Let

$$\left.\begin{aligned} a_2 z^2 + a_3 p_z^2 &= a_2 r^2 \\ \sin\theta &= \frac{\sqrt{a_2}\, z}{\sqrt{a_2 z^2 + a_3 p_z^2}} \end{aligned}\right\} \text{ with the inverse transformations } \left\{\begin{aligned} z &= r \sin\theta \\ \sqrt{\frac{a_3}{a_2}}\, p_z &= r\cos\theta \end{aligned}\right.$$

The z, p_z integration of eq. 8, now becomes using eq. 9,

$$I_z = \int \psi_{0z} f'_{1n}\, dz\, dp_z$$

$$= \frac{N}{a_0} \iint \left[1 + \frac{b^2}{4 a_2 a_3} \left(\frac{r^4 \theta^2}{2} + r^2 \sin^2\theta \cdot r^2 \cos^2\theta - r^2 \cdot r \sin\theta \cdot r\cos\theta \cdot \theta \right) \right] r\, dr\, d\theta \sqrt{\frac{a_2}{a_3}} \qquad \text{H-11}$$

$$= \frac{1}{a_0} \left(1 + I'_z \right)$$

where

$$I'_z = \frac{b^2}{4 a_2 a_3} N \sqrt{\frac{a_2}{a_3}} \int_0^{\rho} r^5 dr \cdot 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\theta^2}{2} + \frac{\sin^2\theta \cos^2\theta}{2} - \frac{\theta \sin 2\theta}{2} \right] d\theta$$

$$= \frac{b^2 N}{4 a_2 a_3} \sqrt{\frac{a_2}{a_3}}\, \frac{\rho^6}{6} \cdot 2 \left[\frac{2}{2} \frac{\left(\frac{\pi}{2}\right)^3}{3} + \frac{2}{2} \cdot \frac{1}{8} \cdot \frac{\pi}{2} - \frac{2}{8} \cdot \pi \right] \qquad \text{H-12}$$

$$= \frac{b^2}{a_2 a_3} \frac{\rho^4}{12} \left(\frac{\pi^2}{2 \cdot 12} - \frac{6}{32} \right)$$

$$= -\Omega^2 h \frac{\rho^4}{r_0^4} \frac{1}{48} \left(\frac{\pi^2}{24} - \frac{3}{16} \right) = -\Omega^2 h \frac{\rho^4}{r_0^4} (.0047)$$

and

$$I_z = \frac{1}{a_0} \left[1 - \Omega^2 h \frac{\rho^4}{r_0^4} (.005) \right]. \qquad (27)$$

In the integration over θ we have used Pierce #176, 201. We have also used the values of b, a_2, a_3 from eq. 2-55. The only difference between eq. 8 and the dispersion relation of eq. 2-39a, is that now the number of particles is multiplied by a factor

$$\alpha_z \equiv 1 - \Omega^2 h \frac{\rho^4}{r_0^4} (.005) \qquad \text{H-13}$$

Thus the effect of z betatron oscillations is to improve stability, as expected, but only by a very small amount. Note that even though this effect goes as l^2, it cannot be large for small wavelengths, because the equation for the potential (C-2) and hence eq. 8 is valid only if $\frac{r_0}{l} \gg \rho$

We have assumed above that $|a_0| \ll \sqrt{a_2 a_3}$ to make E in eq. 4 small so that $e^{\epsilon} \approx 1 + \epsilon$. By eq. 2-55 this inequality implies that

$$|\Omega - l\dot{\theta}_0 + lk\rho_0| \ll \sqrt{n}\,\dot{\theta}_0 \qquad \text{H-14}$$

since

$$|\Omega - l\dot{\theta}_0 + lk\rho_0| \leq |\Omega - l\dot{\theta}_0| + |lk\rho_0| \qquad \text{H-15}$$

Eq. 14 is valid if

$$|\Omega - l\dot{\theta}_0| + |lk\rho_0| \ll \sqrt{n}\,\dot{\theta}_0$$

or

$$\left|\frac{\Omega - l\dot{\theta}_0}{\sqrt{n}\,\dot{\theta}_0}\right| \ll 1, \qquad \left|\frac{lk\rho_0}{\sqrt{n}\,\dot{\theta}_0}\right| \ll 1 \qquad \text{H-16}$$

This requirement is net for weak currents and small temperature spreads in the beam. See, for example, eqs. 2-8 and G-3.

APPENDIX IX

Betatron r Oscillations

Here we solve equation 2-62 following the same general procedure as used in Appendix VIII.

By standard techniques we must first solve

$$\frac{dp_r}{-a_2 x} = \frac{dx}{a_3 p_r} = \frac{d\psi^{1n}}{a_1 \varphi^{1n} \frac{\partial \psi_0}{\partial p_0} - (a_0 + bx)\psi^{1n}} \qquad \text{I-1}$$

The solution of the equation formed with the first equality is $a_2 x^2 + a_3 p_r^2 = c_1$. Using this to eliminate p_r from the last equality, we obtain a differential equation like eq. H-2 but with bx instead of bz^2. Thus

$$\frac{d\psi^{1n}}{dx} + \frac{(a_0 + bx)\psi^{1n}}{\sqrt{a_3(c_1 - a_2 x^2)}} = \frac{a_1 \varphi^{1n} \frac{\partial \psi_0}{\partial p_0}}{\sqrt{a_3(c_1 - a_2 x^2)}} \qquad \text{I-2}$$

The solution of eq. 2 is:

$$\psi^{1n} = a_1 \varphi^{1n} e^{-\int \frac{a_0 + bx}{\sqrt{a_3(c_1 - a_2 x^2)}} dx} \left[\int \frac{\partial \psi_0}{\partial p_0} e^{\int \frac{a_0 + bx}{\sqrt{a_3(c_1 - a_2 x^2)}} dx} dx \right] \qquad \text{I-3}$$

This is the particular solution. The solution of the homogenous equation again implies that $\varphi^{1n} = 0$, which does not contribute to the N.M.I., but is a trivial radial pulsation.

This solution is similar to eq. H-3, but since $x = f(p_0)$, $\frac{\partial \psi_0}{\partial p_0}$ is also a function of x and thus cannot be taken out of the integral signs, i.e. by eqs. 2-61, 63 and 65

$$\frac{\partial \psi_0}{\partial p_0} = \frac{\partial A_p}{\partial p_0} \cdot f + A_p \frac{\partial f}{\partial (\ )} \cdot 2 a_2 x \left(\frac{-1}{m v_0 (1-h)} \right) \qquad \text{I-4}$$

$$= \frac{\partial A_p}{\partial p_0} \cdot f(c_1) + A_p \frac{\partial f(c_1)}{\partial c_1} \left(\frac{-2 a_2 x}{m v_0 (1-h)} \right)$$

We have reinserted the constant c_1, after the differentiation.

Using Pierce[27] Nos. 121, 129, we can integrate the exponent in eq. 3 to get:

$$E \equiv \int \frac{a_0 + b x}{\sqrt{a_3 (c_1 - a_2 x^2)}} dx = \frac{a_0}{\sqrt{a_2 a_3}} \sin^{-1} \sqrt{\frac{a_2}{c_1}}\, x - \frac{b \sqrt{a_3 (c_1 - a_2 x^2)}}{a_2 a_3} \qquad \text{I-5}$$

We will now show that eq. 3 gives the correct limit, eq. 2-39, for the dispersion relation as the radial betatron oscillation amplitude $\rho \to 0$. From eq. 2-63, $a_2 a_3 = (1-h)\dot{\theta}_0^2$. From the definition of c_1, max $c_1 \approx a_2 \rho^2$. Thus $\frac{b c_1}{a_2} \approx b\rho^2 \approx 2(1-h)\dot{\theta}\frac{\rho^2}{r^2}$ and the second term in E is of order $\frac{\rho^2}{r^2}$. The first term remains finite as $\sin^{-1}$ is $\approx$ unity, and $|a_0| = \Omega - 2\dot{\theta}_0 + 2 k p_0$ is independent of x and p_r. Thus the b term in E goes to zero and we can get the appropriate limit by setting b = 0 in eq. 3. The integrations, using eq. 4, are straightforward and may be done in closed form. Note that the first term in eq. 4 is independent of x, while the second is $\sim$ x. Thus the following two integrals are required for doing the integrations of eq. 3:

$$\int \frac{e^{\int \frac{a_0}{\sqrt{\ }} dx}}{\sqrt{\ }} dx = \frac{1}{a_0} e^{\int \frac{a_0}{\sqrt{\ }} dx} = \frac{1}{a_0} e^{\frac{a_0}{\sqrt{a_2 a_3}} \sin^{-1} \sqrt{\frac{a_2}{c_1}} x}$$

I-6

$$\int \frac{x \, e^{\int \frac{a_0}{\sqrt{\ }} dx}}{\sqrt{\ }} dx = \frac{a_0 x - \sqrt{a_3 (c_1 - a_2 x^2)}}{a_0^2 + a_2 a_3} e^{\int \frac{a_0}{\sqrt{\ }} dx} .$$

We have again abbreviated $\sqrt{\ } \equiv \sqrt{a_1(c_1 - a_2 x^2)}$. Now that the integrations have been performed over dx, the p_r^2, x^2 dependence of c_1, is again inserted into eq. 3. Thus $\sqrt{\ } = a_3 p_r$ and eq. 3 becomes

$$\psi_{1n} = a_1 \varphi_{1n} \left[\frac{\partial A}{\partial p_0} \cdot f \cdot \frac{1}{a_0} + A \frac{\partial f}{\partial c_1} \cdot \frac{-2a_2}{h_1 v_0 (1-h)} \cdot \frac{(a_0 x - a_3 p_r)}{a_0^2 + a_2 a_3} \right] .$$ I-7

That this equation satisfies eq. 2-62, with $x \approx 0$, i.e. b = 0, is easily verified by substitution. The second term of eq. 7 will be shown presently to give zero when integrated over in the dispersion relation. To obtain the dispersion relation ψ_{1n} must be inserted into eq. C-2 written in terms of the fourier components. We ignore the $dz \, dp_z$ integration or if we wish, assume that ψ_{1n} contains a factor normalized to one on the field z, p_z. The dispersion relation is now, after cancelling φ_{1n},

$$1 = \frac{eq}{r_0} \int \frac{\psi_{1n}}{\varphi_{1n}} dp_0 \, dr \, dp_r .$$ I-8

For ease of integration we will define $f(a_2 X^2 + a_3 p_r^2)$ where $X \equiv r - r_e$ as

$$f = \begin{cases} N & c_1 = a_2 (r-r_e)^2 + a_3 p_r^2 \le a_2 \rho^2 \\ & \text{for} \\ 0 & c_1 > a_2 \rho^2 \end{cases}$$

N is a normalization constant, and just as in the discussion following eq. H-10, $N = \frac{1}{A_e}$, where $A_e = \pi \sqrt{\frac{a_2}{a_3}} \rho^2$. Note that since $f = f(p_0)$ the $dr\, dp_r$ integration must be done first. The first term gives

$$I = \frac{e g a_0}{r_0} \int \frac{\partial A_y}{\partial p_0} \cdot \frac{1}{a_0}\, dp_0 \qquad \text{I-9}$$

which is the same dispersion relation as eq. 2-39. Integration of the second term of ψ_{1n} is facilitated by going over to polar coordinates. The procedure is exactly the same as used in the discussion following eq. H-10. We define

$$a_2 X^2 + a_3 p_r^2 = a_2 R^2 \qquad \text{with the inverse transformations} \qquad X = R \sin\theta$$

$$\sin\theta = \frac{\sqrt{a_2}\, X}{\sqrt{a_2 X^2 + a_3 p_r^2}} \qquad \sqrt{\frac{a_3}{a_2}}\, p_r = R \cos\theta$$

We can now write

$$\frac{\partial f}{\partial c_1} = \frac{\partial f}{\partial R} \frac{\partial R}{\partial c_1} = \frac{1}{a_2 R} \frac{\partial f}{\partial R} = N \frac{[\delta(R) - \delta(R-\rho)]}{2 a_2 R} \qquad \text{I-10}$$

as f may be regarded as a pulse function which is constant as R goes from 0 to ρ and is zero otherwise. Also $X = r - r_0 - \frac{p_0}{m v_0 (1-n)}$ and in the r integration, p_0 is a constant. Thus evaluating the Jacobian

$$dr\, dp_r = \sqrt{\frac{a_2}{a_3}}\, R\, dR\, d\theta$$

and $\int \Psi_{1n}^{\pi}\, dr\, dp_r$

$$= a_1 \varphi_{1n} A_p \int \frac{N\left[\delta(R) - \delta(R-\rho)\right]}{(a_0^2 + a_2 a_3) R\, m v_0 (1-n)} \left(a_3 \sqrt{\frac{a_2}{a_3}} R \cos\theta - a_0 R \sin\theta\right) R\, dR \sqrt{\frac{a_2}{a_3}}\, d\theta$$

$$= a_1 \varphi_{1n} \frac{1}{\pi \rho\, m v_0 (1-n)} \int \frac{\left(\sqrt{a_2 a_3} \cos\theta - a_0 \sin\theta\right)}{a_0^2 + a_2 a_3}\, d\theta \qquad \text{I-11}$$

We note now that p_r is defined only for positive values. To find the correct form of Ψ_{1n} for negative p_r, we must go back to eq. 3 and change the sign of the square root term. In eq. 7 this results only in p_r changing sign. Thus the sin and cos representations for x and p_r are valid as θ goes through 2π and the integral in eq. 11 gives zero so that the dispersion relation is given by eq. 9. Thus eq. 7 gives the correct dispersion relation, as asserted.

It is interesting to speculate on the significance of the δ functions in the second term of Ψ_{1n}. Owing to changes of p_θ of the particles, the radial position is displaced. This occurs for each of the tubes of particles centered on different values of p_θ. The distribution function of a tube slightly shifted differs by a δ-function from the previous distribution. This distribution is $\sim \delta \cos\theta$, where the δ-function is on the surface, i.e. $\delta(R-\rho)$, for a tube of radius ρ. This shift however does not affect the density of particles and hence does not contribute to the dispersion relation of an effect, which occurs because of θ electric fields.

We shall now derive the correction to the above results due to finite betatron oscillation amplitude. Since both terms in the exponent of eq. 3 are small we shall now expand the exponential in the same manner as we did for the z oscillations and then integrate. We write $e^{E} \approx 1 + E + \frac{E^2}{2}$ and integrate in eq. 3. We shall first consider the first term in $\frac{\partial \psi_0}{\partial p_0}$ which is independent of x. Thus using eq. 5:

$$\int \frac{\left(1 + E + \frac{E^2}{2}\right) dx}{\sqrt{a_3(c_1 - a_2 x^2)}} = \frac{1}{\sqrt{a_2 a_3}} \sin^{-1} \sqrt{\frac{a_2}{c_1}}\, x \qquad (\int \cdot 1)$$

$$+ \frac{a_0}{2 a_2 a_3} \left[\sin^{-1} \sqrt{\frac{a_2}{c_1}}\, x\right]^2 - \frac{b x}{a_2 a_3} \qquad (\int \cdot E) \qquad \text{I-12}$$

$$+ \frac{a_0^2 \left[\sin^{-1}\right]^3}{2 \cdot 3 \cdot (a_2 a_3)^{3/2}} - \frac{a_0 b}{(a_2 a_3)^{3/2}} \left[x \sin^{-1} + \sqrt{\frac{c_1}{a_2}\left(1 - \frac{a_2 x^2}{c_1}\right)}\right]$$

$$+ \frac{b^2}{(2 a_2 a_3)^2} \left(x \sqrt{a_3(c_1 - a_2 x^2)} + c_1 \sqrt{\frac{a_3}{a_2}} \sin^{-1}\right) \qquad \left(\int \cdot \frac{E^2}{2}\right)$$

This result differs from the integral in the bracket of eq. 3 by a constant. The argument parallels that following eq. H-6. If we set x = 0 in eq. 12 we obtain zero, whereas from eq. 6 we have $\frac{1}{a_0}$. Thus the value of the constant is found.

Writing now

$$\psi_{1n}^{I} = a_1 \varphi_{1n} \frac{\partial \Lambda}{\partial p_0} f(c_1) f_{1n}^{I} \qquad \text{I-13}$$

for the part of ψ_{1n} due to the first term of $\frac{\partial \psi_0}{\partial p_0}$, we have

$$f_{1n}^{I} \approx \left(1 - E + \frac{E^2}{2} - \frac{E^3}{3}\right)\left(\frac{1}{a_0} + \int \frac{\left(1 + E + \frac{E^2}{2}\right) dx}{\sqrt{}}\right)$$

$$\approx \left(1 - E + \frac{E^2}{2} - \frac{E^3}{3}\right)\frac{1}{a_0} + \left(1 - E + \frac{E^2}{2}\right)\int \frac{1 \cdot dx}{\sqrt{}} + (1 - E)\int \frac{E\, dx}{\sqrt{}} + 1 \cdot \int \frac{E^2 dx}{2\sqrt{}}$$

$$= \frac{1}{a_0} + (1 - 1)\frac{\sin^{-1}}{\sqrt{a_2 a_3}} + \frac{b\sqrt{}}{a_0 a_2 a_3} + \left(\frac{a_0}{2} - a_0 + \frac{a_0}{2}\right)\frac{(\sin^{-1})^2}{a_2 a_3} \qquad \text{I-14}$$

$$+\left(\frac{-2a_0}{2a_0}+1\right)\frac{b\sqrt{}\,\sin^{-1}}{(a_2 a_3)^{3/2}}+\frac{1}{a_0}\,\frac{b^2(\sqrt{})^2}{2\,(a_2 a_3)^2}-\frac{b\,x}{a_2 a_3}$$

$$+\left(\frac{-a_0^3}{a_0\,3!}+\frac{a_0^2}{2}-\frac{a_0^2}{2}+\frac{a_0^2}{3\cdot 2}\right)\frac{(\sin^{-1})^3}{(a_2 a_3)^{3/2}}+\left(\frac{3a_0}{3!}-a_0+\frac{a_0}{2}\right)\frac{b\sqrt{}\,(\sin^{-1})^2}{(a_2 a_3)^2}$$

$$+\left(\frac{-3}{3!}+\frac{1}{2}\right)\frac{b^2(\sqrt{})^2\sin^{-1}}{(a_2 a_3)^{3/2}}+\frac{1}{3!\,a_0}\left[\frac{b(\sqrt{})}{a_2 a_3}\right]^3+\frac{(1-1)\,a_0\,b\,(\sin^{-1})\,x}{(a_2 a_3)^{3/2}}$$

$$+\left(\frac{1}{4}-1\right)\frac{b^2\sqrt{}\,x}{(a_2 a_3)^2}+\sim\sqrt{1-\frac{a_2}{c_1}x^2}+\sim\sin^{-1}$$

In the foregoing calculation the last three lines are the result of going to one higher order of E than in eq. H-7. We have calculated terms up to $\frac{\rho^2}{a_0}$, $a_0\rho$. Those terms which are odd in p_r or x will not contribute to the dispersion relation, eq. 8. Thus the only terms in eq. 14 which contribute are:

$$f^{I}_{1n} = \frac{1}{a_0}\cdot\left[1+\frac{b^2(a_3\,p_r)^2}{2\,(a_2 a_3)^2}\right] \qquad \text{I-15}$$

Next we must calculate the contribution ψ^{II}_{1n} i.e., the part of ψ_{1n} due to the second term of $\frac{\partial\psi_s}{\partial p_o}$ in eq. 4 which is proportional to x. We again evaluate the bracket of eq. 3 by expanding the exponent and writing $e^{E}\approx 1+E+\frac{E^2}{2}$. Thus we need the following expression:

$$\int\frac{x\left(1+E+\frac{E^2}{2}\right)dx}{\sqrt{}} = -\frac{\sqrt{}}{a_2 a_3} \qquad \text{J-1}$$

$$- \frac{a_0 \sqrt{\quad}\, \sin^{-1}}{(a_2 a_3)^{3/2}} + \frac{a_0 x}{a_2 a_3} - \frac{b x^2}{2 a_2 a_3} \qquad \int \cdot E$$

$$- \frac{\sqrt{\quad}}{a_2 a_3} \frac{E^2}{2} + \frac{a_0^2}{(a_2 a_3)^{3/2}} \left(x \sin^{-1} + \sqrt{\frac{c_1}{a_2} - x^2} \right) + \frac{a_0(-b)}{(a_2 a_3)^2} \cdot \frac{1}{2} \left(x \sqrt{\quad} + c_1 \sqrt{\frac{a_3}{a_2}} \sin^{-1} \right)$$

$$+ \frac{b a_0}{(a_2 a_3)^{3/2}} \frac{c_1}{a_2} \frac{1}{4} \left[\left(\frac{2 a_2 x^2}{c_1} - 1 \right) \sin^{-1} + \sqrt{\frac{a_2}{c_1}}\, x \sqrt{1 - \frac{a_2 x^2}{c_1}} \right] \qquad \int \cdot \frac{E^2}{2} \qquad \text{I-16}$$

$$+ \frac{b^2}{3} \frac{(\sqrt{\quad})^3}{(a_2 a_3)^3}$$

where as usual $\sin^{-1} \equiv \sin^{-1} \sqrt{\frac{a_2}{c_1}}\, x$, $\sqrt{\quad} \equiv \sqrt{a_3 (c_1 - a_2 x^2)}$.

If we now write

$$\psi_{1n}^{II} = a_1 \varphi_{1n} A_p \frac{\partial f}{\partial c_1} \cdot \frac{-2 a_2}{m_1 v_0 (1-n)} \cdot f_{1n}^{II} \qquad \text{I-17}$$

then

$$f_{1n}^{II} = \left(1 - E + \frac{E^2}{2} \right) \cdot \int \frac{x \left(1 + E + \frac{E^2}{2} \right)}{\sqrt{\quad}}\, dx \qquad \text{I-18}$$

(Note that eq. 17 reduces to the second term in eq. 7, for small x, p_r , and a_0 .)

Instead of writing out the expression f_{1n}^{II} in full, we will note only those terms which give a non-zero contribution to the dispersion relation, eq. 8. From eq. 10 we note that $\frac{\partial f}{\partial c_1}$ is even in x and p_r . Thus only terms even in x and p_r will contribute. Replacing c_1 by $a_2 x^2 + a_3 p_r^2$ in eqs. 5 and 16 gives the p_r dependence. Since all the square roots are positive, the equations are valid only for positive p_r . Thus in eq. 3 we must substitute $-\sqrt{\quad}$ for $\sqrt{\quad}$ to find the behavior for the negative p_r . We find that the $\sqrt{\quad}$ terms change sign and so does $\sin^{-1}\sqrt{\frac{a_2}{c_1}}x$. Thus the $\sqrt{\ }$ and $\sin^{-1}$ terms are odd in p_r , while $\sin^{-1}$ is also odd in x. Thus terms like the following give a zero contribution to the dispersion relation, $p_r \sin^{-1}x$, x, $p_r(\sin^{-1}x)^2$, $p_r^2\sin^{-1}x$, p_r^3, $x\sin^{-1}x$, p_r, $x p_r$ etc. Finite contributions are obtained from terms like x^2, p_r^2, $(\sin^{-1})^2$ and $x(\sin^{-1}x)p_r$.

The expression E is of order a_0, ρ . We thus find that

$\int \frac{x}{\sqrt{\ }}dx$ is of order ρ

$\int \frac{xE}{\sqrt{\ }}dx$ is of order $a_0\rho$, ρ^2

$\frac{1}{2}\int \frac{xE^2}{\sqrt{\ }}dx$ is of order $a_0^2\rho$, $a_0\rho^2$, ρ^3

All terms of order $a_0^n\rho$, $(n = 0,1,2\ldots)$ in f_{2n}^{II} must be odd and integrate to zero, because they must be compounded of the factors $(\sin^{-1}x)^n x$ or $(\sin^{-1}x)\,p_r$. Thus the lowest order non-zero term is of order ρ^2 . There are also terms of order $a_0^2\rho^2$, $a_0^4\rho^4$ etc., which we will neglect because we assume $a_0 \ll 1$. Thus to terms of order ρ^2

$$f_{1n}^{II} = 1\cdot\int\frac{x(1+E)dx}{\sqrt{\ }} - E\cdot\int\frac{x}{\sqrt{\ }}dx$$

and the non-vanishing terms are:

$$f_{1n}^{II} = -\frac{bx^2}{2a_2a_3} - \frac{b\,(\sqrt{\ })^2}{(a_2a_3)^2}$$

$$= -\frac{b}{2a_2^2a_3}\left(a_2x^2 + 2a_3p_r^2\right) \qquad \text{I-19}$$

$$\equiv \frac{-b}{2a_2^2a_3}\cdot d_1$$

Collecting now the terms in eqs. 13, 15, 17 and 19 and inserting them into the dispersion relation, eq. 8, we obtain

$$I = \frac{eg}{r_0}\iint\left\{a_1 \frac{\partial A_p}{\partial p_\theta} f(c_1) \frac{1}{a_0}\left(1 + \frac{b^2}{2}\frac{a_3\, p_r}{a_2\, a_3}\right)\right. \tag{I-20}$$

$$\left. + a_1 A_p \frac{\partial f}{\partial c_1}\frac{(-2a_2)}{m v_0(1-n)}\frac{(-b d_1)}{2a_2^2 a_3}\right\} dp_\theta\, dr\, dp_r$$

We shall do the $dr\,dp_r$ integration first. Thus p_θ is a constant and $dr = dx$. The first factor is

$$I_1 = \iint f(c_1)\,dx\,dp_r = 1$$

using the definition of 'f' given below eq. 8. The next factor is

$$I_2 = \iint f(c_1)\,p_r^2\,dx\,dp_r$$

$$= \sqrt{\frac{a_3}{a_2}}\,\frac{1}{\pi\rho^2}\int_0^{\rho}\frac{a_2}{a_3}R^2\int_0^{2\pi}\cos^2\theta\; R\,dR\sqrt{\frac{a_2}{a_3}}\,d\theta$$

$$= \frac{a_2}{a_3}\frac{\rho^2}{4}$$

We have evaluated the integral by changing to polar coordinates as defined in the discussion following eq. 8. Finally, we integrate the last factor, using polar coordinates again, and eq. 10. Thus,

$$I_3 = \iint \frac{\partial f}{\partial c_1}\,d_1\,dx\,dp_r$$

$$= \sqrt{\frac{a_3}{a_2}}\,\frac{1}{\pi\rho^2}\int_0^{\rho}\frac{\left[\delta(R) - \delta(R-\rho)\right]}{2a_2 R}\cdot a_2 R^2\int_0^{2\pi}(1+\cos^2\theta)\,R\,dR\sqrt{\frac{a_2}{a_3}}\,d\theta$$

$$= -\frac{3}{2}$$

Equation 20 now becomes, using I_1, I_2, and I_3,

$$I = \frac{e g a_1}{r_0}\int\left\{\frac{\partial A_p}{\partial p_\theta}\cdot\frac{1}{a_0}\left(1 + \frac{b^2\rho^2}{8 a_2 a_3}\right) - \frac{3}{2}\frac{A_p\, b}{m v_0(1-n)a_2 a_3}\right\} dp_\theta$$

A_p is normalized to N on the field θ, p_θ. Defining $A_p = \frac{N}{2\pi}\psi_0$, where ψ_0 is normalized to one on the field p_θ, and inserting the values of a_i and b from eq. 2-63, we obtain

$$I = \frac{e^2 g N}{2\pi r_0} \int \frac{\partial \psi_0}{\partial p_0} \frac{1 - \frac{\eta^2}{8(1-n)} \frac{\rho^2}{r^2}}{\dot{\theta}_0 - \frac{\Omega}{\ell} - k p_0} d p_0 - \frac{e^2 g N}{2\pi r_0} \cdot \frac{3}{2} \cdot \frac{\eta^2}{m_0 r_0^2 \dot{\theta}_0^2 (1-n)^2}$$

Writing $\nu = \frac{e^2}{m_0 c^2} \cdot \frac{N}{2\pi r_0}$ we obtain

$$I = \frac{\nu g m_0 c^2 \left(1 - \frac{\eta^2}{8(1-n)} \frac{\rho^2}{r_0^2}\right)}{1 + \nu g \eta^2 \cdot \frac{3}{2} \frac{c^2}{r_0^2} \frac{1}{\dot{\theta}_0^2 (1-n)^2}} \int \frac{\partial \psi_0}{\partial p_0} \frac{d p_0}{\dot{\theta}_0 - \frac{\Omega}{\ell} - k p_0} \qquad \text{I-21}$$

If we use a pulse function for ψ_0, the integration may be performed as in Appendix VII. Because of the factor before the integral in eq. 21, we obtain:

$$\left(\frac{\Omega}{\ell} - \dot{\theta}_0\right)^2 = (\Delta\dot{\theta})^2 - \nu g \frac{c^2}{r^2} \left[\frac{1 - \frac{\eta^2}{8(1-n)} \frac{\rho^2}{r_0^2}}{1 + \frac{\nu g c^2}{r_0^2} \cdot \frac{3}{2} \cdot \frac{\eta^2}{\dot{\theta}_0^2 (1-n)^2}} \right] \qquad \text{I-22}$$

instead of eq. G-3. Thus the betatron oscillations have two effects. One is to decrease the effective number of particles, because of the ρ^2 term. The other shows that the maximum rise time of the instability is of the order of the radial oscillation frequency. In the case of the z oscillations, the effective number of particles was decreased by a term $\sim \rho^4$. Note too that $\frac{|\eta| \rho}{r_0} \ll 1$ in order that eq. C-2 holds. Thus this factor is always small and the finite radial oscillations have negligible stabilizing effect.

We now show how the additional term in the denominator affects the rise time. The dispersion relation, eq. 21, is valid only if $\Delta\dot{\theta} \ll \dot{\theta}_0$, because we have assumed a thin beam. For stability $(\Delta\dot{\theta})^2 > \frac{\nu g c^2}{r_0^2}$ and hence for small η, the second term in the denominator is small.

It can be large only when there is instability. In the limit when $\nu g \frac{c^2}{r_0^2} \gg \frac{\dot{\theta}_0^2(1-h)^2}{l^2}$ we see that the instability growth rate is

$$\frac{\Omega}{l} - \dot{\theta}_0 \approx \pm i \frac{\dot{\theta}_0}{l}(1-h)$$

These results are all reasonable. We expect the betatron oscillations to improve the stability. It is, however, a surprise that the effect is so small. That eq. G-3 is invalid for large growth rates and becomes eq. 22 is also reasonable because the constraint equation, eq. 2-6, breaks down for large growth rates. Note, however, that in writing eq. 19 we assumed that higher powers of $\frac{a_0}{\sqrt{a_2 a_3}}$ are negligible. This requires the validity of eq. H-16, with $h \to 1-h$. For the unstable case, as $\Delta\dot{\theta} \ll \dot{\theta}_0$, this implies that the second term in the denominator of eq. 22 is small. Thus our conclusions above about the limiting growth rate which obtains when this term is large are not accurate. It seems likely, however, that the inclusion of the higher order terms will not change the qualitative result.

APPENDIX X

Infinite Beams

We shall show here that an equation obtained from Bludman et al.[19] is the same as one obtained from eq. 3-44.

In eq. 3-44 let ψ'_{o+} represent a pulse function of very narrow width and let $V_1 = 0$. Then the integration of the first integral in eq. 3-44 is done as in Appendix VII. If the width of the pulse now goes to zero, then this integral is $\frac{k_1^2}{\Omega^2}$, as ψ_{o+} in eq. 3-44 is normalized to one on the field V_+. Thus eq. 3-44 may be written as

$$1 = \frac{\omega_{p+}^2}{\omega^2} + \frac{\omega_{p-}^2}{\gamma^3 k_1^2} \int \frac{\frac{\partial f_o}{\partial v}}{v - \frac{\omega}{k_1}} dv \qquad \text{J-1}$$

which describes the longitudinal oscillations of a relativistic electron beam, which is not too hot, travelling through a stationary cold ion background. We have also made the change of variables $v = v_- + V_2$ in the second integral of eq. 3-44.

Now we shall obtain eq. 1 from Bludman et al.[19] From eq. 2.19, p. 750[19]

$$\omega^2 = \omega_p^2 + I_{2+} \quad .$$

ω_p^2 represents the plasma frequency of a cold background plasma. This equation is equally true if the background has no electrons but only ions. Then $\omega_p^2 \rightarrow \omega_{p+}^2$, and

$$\omega^2 = \omega_{p+}^2 + I_{2+} \quad . \qquad \text{J-2}$$

By eq. 2.2a,[19]

$$I_{zz} = \int \frac{4\pi e^2 f_{00}}{\Omega - V_z k_z} v \frac{V_+}{\theta_{\parallel}} d^3v \qquad \text{J-3}$$

where we have made the changes $V_0 + V_z \to v$, $d^3 v \to d^3 v$. If we assume no perpendicular velocity components then $V_\perp = 0$, and from the equation on the top right of p. 748,[19]

$$\nabla_p f_{00} = \frac{-V_{\parallel}}{\theta_{\parallel}} f_{00} \qquad \text{J-4}$$

and

$$\nabla_p f_{00} = \frac{\partial f_{00}}{\partial p_z} = \frac{\partial v}{\partial p} \frac{\partial f_{00}}{\partial v} = \frac{1}{\gamma^3 m} \frac{\partial f_{00}}{\partial v} . \qquad \text{J-5}$$

f_{00} is normalized to n, the particle density, on the field $d\vec{v}$. Thus inserting eqs. 4 and 5 into eq. 3 and integrating over the perpendicular components, we obtain $(\Omega = \omega - v_0 k_z)$,

$$I_{zz} = -\int \frac{4\pi e^2 n}{\gamma^3 m_0} \frac{v\omega}{\omega - v k_z} \frac{\partial f_0}{\partial v} dv = -\frac{\omega_p^2}{\gamma^3} \int \frac{v\omega \frac{\partial f_0}{\partial v}}{\omega - v k_z} dv \qquad \text{J-6}$$

where f_0 is normalized to one on the field dv. Also

$$\int \frac{v \frac{\partial f_0}{\partial v}}{\omega - v k_z} dv = \int \frac{v \frac{\partial f_0}{\partial v}}{\omega - v k_z} dv + \frac{1}{k_z} \int \frac{\partial f_0}{\partial v} dv \qquad \text{J-7}$$

$$= \int \frac{\left(v + \frac{\omega}{k_z} - v\right) \frac{\partial f_0}{\partial v}}{\omega - v k_z} dv = \frac{\omega}{k_z^2} \int \frac{\frac{\partial f_0}{\partial v}}{\frac{\omega}{k_z} - v} dv .$$

Inserting this result into eq. 6, and then eq. 6 into eq. 2 gives eq. 1

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APPENDIX XI

Plasma Oscillations and Lorentz Transformations

We will show that the dispersion relation for longitudinal oscillations of a single stream of cold relativistic electrons, as obtained from eqs. 3-44 and 3-36, may be obtained from the equations for a stationary 'stream' by a Lorentz transformation.

First we will consider the case of an infinitely wide beam as it is simpler. Setting the number of ions equal to zero in eq. 3-44, gives $\omega_{p+} = 0$. Using next a pulse function for ψ_{0-} as done in Appendix VII, and then setting the temperature term equal to zero in eq. 3-44, the dispersion relation for a relativistic beam becomes,

$$\Omega = V_- k \pm \left(\frac{1}{\gamma^3} \omega_{p-}^2 \right)^{1/2} \qquad \text{K-1}$$

In a stationary system, i.e., moving with the beam, we see the longitudinal plasma oscillations, given by (28)

$$\Omega = \pm \omega_{po}$$

where

$$(\omega_{po})^2 = \frac{4\pi n_0 e^2}{m_0} \qquad \text{K-2}$$

n_0 is the electron density measured in the beam system and m_0 is the mass measured in that system. The wave disturbance in the plasma is represented by $e^{i(\omega t - kx)}$. Thus ($\vec{k}$, $i\frac{\omega}{c}$) form a four-vector. The beam system is unprimed, while in the lab system, moving with a velocity -V with respect to the beam, the quantities are primed. Thus

$$\omega = \gamma (\omega' - V k')$$

$$\text{or} \quad \omega' = \frac{\omega}{\gamma} + Vk' \qquad \text{because } v = -V. \qquad \text{K-3}$$

In the beam system, ω is given by Ω of eq. 2 or

$$\omega = \Omega = \pm \left(\frac{4\pi n_0 e^2}{m_0} \right)^{1/2}. \qquad \text{K-4}$$

We now wish the value of n_0 as measured in the lab system. Since n forms part of a four-vector we have the following transformation equation, where again the primed quantity is in the lab system,

$$n' = \gamma \left(n + V j \right) .$$

The current j is evidently zero in the beam system. Hence if $n = n_0$, $n' = \gamma n_0$, and by eq. 4

$$\omega = \pm \left(\frac{4\pi n' e^2}{\gamma m_0} \right)^{1/2} \qquad \text{K-5}$$

and therefore eq. 3 is

$$\omega' = Vk' \pm \left(\frac{4\pi n' e^2}{\gamma^3 m_0} \right)^{1/2}. \qquad \text{K-6}$$

This is now identical with eq. 1 if the appropriate correspondences are made, including $\omega_{p} = \left(\frac{4\pi n' e^2}{m_0} \right)^{1/2}$ where n' is measured in the lab system as is evident from the article by Bludman et al.[(19)]

Next we proceed to the case of very narrow beams, i.e., $bk \ll 1$ (b = beam radius). As shown by Sturrock[(29)] this modifies the plasma frequency given by eq. 2. For thin beams

$$\omega_p^2 = \frac{4\pi n e^2}{m} \, b^2 k^2 \, \ln\left(\frac{1}{bk} - c\right) \cdot \frac{1}{2}$$

$$\equiv \omega_{po}^2 \cdot b^2 k^2 \, g_s \equiv Q^2 k^2 \qquad (c = .577, \text{ Euler's const.}) \qquad \text{K-7}$$

This is true non-relativistically and hence holds true for a relativistic beam if ω_p is observed in a coordinate system moving with the beam. In the lab system we observe a frequency Ω , which may be derived from eq. 3-36. We set the number density of ions equal to zero, or equivalently their mass infinite, in which case the first term on the R.H.S. of eq. 3-36 is zero. Setting the temperature term equal to zero, now gives

$$\Omega = V_o k' \pm \left(\frac{\nu' g c^2 k'^2}{\gamma^5}\right)^{1/2} \qquad \text{K-8}$$

if $\frac{\nu' g}{\gamma} \ll 1$. Also we have set $\frac{r_o}{l} = k'$, and k' is measured in the lab frame. We now wish to make a transformation from the ω , k system to the ω' , k' system moving with velocity -V. We have

$$\omega' = \gamma(\omega + Vk)$$
$$= \gamma(Q + Vk) \qquad \text{K-9}$$

using also eq. 7. Also

$$k' = \gamma\left(k + \frac{\omega V}{c^2}\right)$$

and $\qquad \text{K-10}$

$$k = \gamma\left(k' - \frac{\omega' V}{c^2}\right) .$$

Substituting this value for k into eq. 9 gives

$$\omega' = \gamma (Q + V) \gamma \left(k' - \omega' \frac{V}{c^2} \right)$$

or

$$\omega' = \frac{(Q+V)k'}{\frac{1}{\gamma^2} + \frac{V^2}{c^2} + \frac{QV}{c^2}} = \frac{(Q+V)k'}{1 + \frac{QV}{c^2}}$$

$$= \frac{Qk' + Vk' - Vk'\left(1 + \frac{QV}{c^2}\right)}{1 + \frac{QV}{c^2}} + Vk' \qquad \text{K-11}$$

$$= \frac{Q\left(1 - \frac{V^2}{c^2}\right)k'}{1 + \frac{QV}{c^2}} + Vk'$$

$$= Vk' + \frac{Qk'}{\gamma^2\left(1 + \frac{QV}{c^2}\right)}$$

using $\frac{1}{\gamma^2} = 1 - \frac{V^2}{c^2}$.

The expression for Q in eq. 7 may be written as

$$Q^2 = \frac{4\pi n e^2}{m_0} b^2 k^2 g_s = 4\pi \frac{N}{2\pi r \cdot \pi b^2} \frac{e^2}{m_0 c^2} c^2 b^2 g_s \qquad \text{K-12}$$

$$= 4 \nu c^2 g_s \quad .$$

This ν is measured in the beam system and is proportional to the density measured in the beam system. Calling this ν_0 , then since by eq. 5, $n' = \gamma n_0$, we have that $\nu' = \gamma \nu_0$. Since the condition for the validity of eq. 8 is $\frac{\nu' g}{\gamma} \ll 1$, the corresponding condition in the beam equation is $\nu_0 g_s \ll 1$. Hence by eq. 12, $Q \ll c$, and eq. 11 becomes after substituting for Q in the numerator and neglecting Q in the denominator,

$$\omega' = Vk' \pm \frac{1}{\gamma^2} \left(\frac{4\nu' c^2 g_s}{\gamma} \right)^{\frac{1}{2}} k' \quad . \qquad \text{K-13}$$

Identifying this ω' with Ω in eq. 8, we see that the equations are identical if

$$4 g_s = g$$

First g_s must be transformed to the lab system. The only quantity in g_s which transforms is k. By eqs. 10 and 13, neglecting the $\frac{V'}{\gamma}$ term which is small

$$k \approx \gamma \left(k' - \frac{V k' V}{c^2} \right) = \frac{k'}{\gamma} \quad .$$

Thus in lab quantities

$$4 g_s = 2 \left(\ln \frac{\gamma}{k' b} - c \right) \qquad \text{K-14}$$

while

$$g = 2 \left(\ln \frac{r_0}{2 b} + .12 \right) \qquad \text{K-15}$$

by eq. C-13.

These constants differ somewhat because one refers to a circular geometry, and the other to a linear geometry, and also there is a factor γ that does not appear in eq. 15. This suggests that eq. 15 is in error, apparently because the retardation terms were neglected in writing eq. C-12. Thus since eq. 14 is valid only for small $\frac{V'}{\gamma}$, the factor γ which should appear in eq. 15 is probably

$$\gamma_{\Omega 2} \equiv \frac{1}{\sqrt{1 - \left(\frac{\Omega r_0}{2 c} \right)^2}} \quad .$$

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DEFINITIONS AND SYMBOLS

a) Roman Letter Quantity Symbols

A^0 = vector potential of the unperturbed beam, $A^0(r, \theta, z, t)$

A^1 = vector potential due to the perturbation, $A^1(r, \theta, z, t)$

A_B = the external betatron potential, defined by eq. A-4

B_o = magnetic field at the equilibrium orbit

C = velocity of light

g_l = $2\left(\ln \frac{r_o}{|l|\rho} + .12\right)$ for small l, $\frac{r_o}{|l|\rho} \gg 1$, and $l \neq 0$; for $l = 0$ write l =.14. Its value is $\approx$ 5. For larger l it is given by eqs. C-19, C-12 and C-13.

$g \equiv g_l$

$\hat{g}_l = g_l + \frac{1}{l^2}$. For l = 1, $\frac{1}{l^2}$ = 1.3, also $\hat{g}_o = g_1$. For more accuracy see eqs. C-19, C-12 and C-13.

$\hat{g} \equiv \hat{g}_l$

$k = \frac{\alpha}{\gamma m_o r_o^2}$, where $\alpha = \frac{1}{1-n} - \frac{1}{\gamma^2}$

$k_z = \frac{2\pi}{\lambda}$, the magnitude of the wave vector when the disturbance is of the form $e^{ik_z z - i\Omega t}$

l = integers, $\pm$ 1, 2, 3..... it gives the spatial dependence of a disturbance through $e^{il\theta - i\Omega t}$

m_o = rest mass

$m_{\pm}$ = rest mass of the positively or negatively charged particle

m = relativistic mass, equals γm_o

n = field index, the exponent in $B_z = B_o \left(\frac{r_o}{r}\right)^n$

N = total number of particles in the beam of either specie

p_r = radial canonical momentum (see Appendix I)

p_θ = canonical angular momentum

p_z = axial canonical momentum

q = electric charge density

r = radial coordinate in a cylindrical lab coordinate system

r_o = equilibrium orbit radius

R = r_o

t = time measured in the lab system

$v_\pm = r_o \dot{\theta}_{o\pm}$, the average velocity of either beam

z = axial coordinate in a cylindrical lab coordinate system

b) <u>Greek Letter Quantity Symbols</u>

$\alpha = \frac{1}{1-n} - \frac{1}{\gamma^2}$, is negative in the negative mass region

$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$, the relativistic γ factor

$\frac{1}{\gamma_g^2} = \frac{1}{\gamma^2} - \frac{v^2}{c^2} \frac{\delta g_1}{g}$, gives the deviation from $\frac{1}{\gamma^2}$ due to the circular geometry

$\delta g_1 = \hat{g}_1 - g_1$, a small positive quantity for $|\eta| > 0$

Δ = range of p_θ values for which a pulse distribution function gives non-zero values

$\Delta v_\pm = r_o \Delta \dot{\theta}_\pm = \frac{r_o}{2} \left(\frac{\alpha \Delta}{\gamma m_o r_o^2} \right)_\pm$, a measure of the beam temperature which contributes to the stabilization of the negative mass instability

$\Delta v_{\parallel\pm} = \frac{r_o}{2} \left(\frac{\Delta}{\gamma^3 m_o r_o^2} \right)_\pm$, a measure of the beam temperature which is effective for pure longitudinal oscillations

ΔR = width of the betatron well (see Appendix I)

$\epsilon_H = (1 - n) \frac{\Delta R}{R}$, for parabolic wells given by eq. A-4, a number which is always less than one

θ = angular coordinate in a cylindrical lab coordinate system

$\dot{\theta}_0 = \frac{-e B_0}{\gamma m_0 c}$, the cyclotron frequency at the equilibrium orbit r_o

$\nu = \frac{N}{2\pi R} \frac{e^2}{m_- c^2}$, Budker's parameter, dimensionless measure of the lineal particle density (ν = 1 gives 17,000 amps if $v = c$)

$\nu_- \equiv \nu$

$\nu_+ = \nu \frac{m_-}{m_+}$

ρ = minor radius of the beam, as shown in Fig. 1

φ^0= electrostatic potential of the unperturbed beam, $\varphi^0 (r, \theta , z, t)$

φ'= potential due to the perturbation, $\varphi' (r, \theta , z, t)$

Ψ_0 = the distribution function of the unperturbed beam, $\Psi_0(r, \theta , z, p_r, p_\theta , p_z, t)$

Ψ_1 = the distribution function of the perturbation, $\Psi_1(r, \theta , z, p_r, p_\theta , p_z, t)$

c) Mathematical Symbols

$\sim$ implies proportional to

$\approx$ implies approximately equal to. This notation follows the SUN Commission's recommendations listed in Physics Today, 15, 19 (1962)

P implies principal value

d) <u>Dictionary</u>

N.M.I. = negative mass instability

Transition Energy= for strong focusing accelerators, 1 - n is replaced by k_s in the expression for α , where $k_s > 1$. The transition energy occurs for such γ than $\alpha = 0$

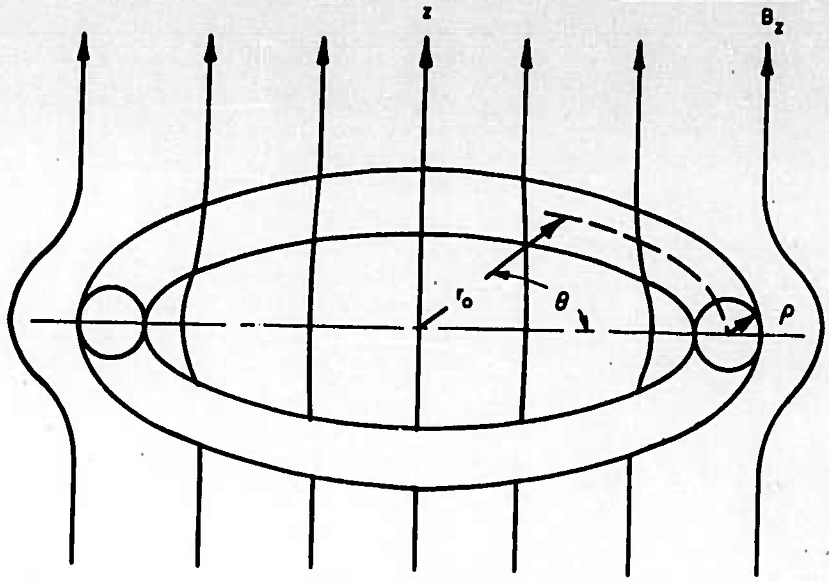

Fig. 1. The physical model. Electrons rotate in the indicated sense.

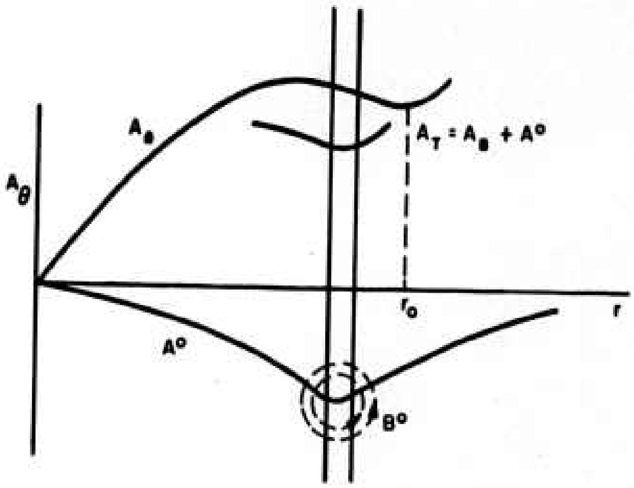

Fig. 2. The vector potential well due to the external field A_B and the self-field $A°$. The lines of the self-field $B°$ are drawn schematically and encircle the torus. (see p 13)

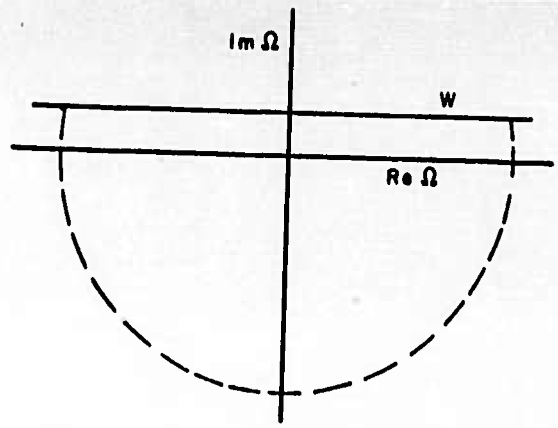

Fig. 3. The line W in the complex Ω plane used for the Laplace transform. (see p 23)

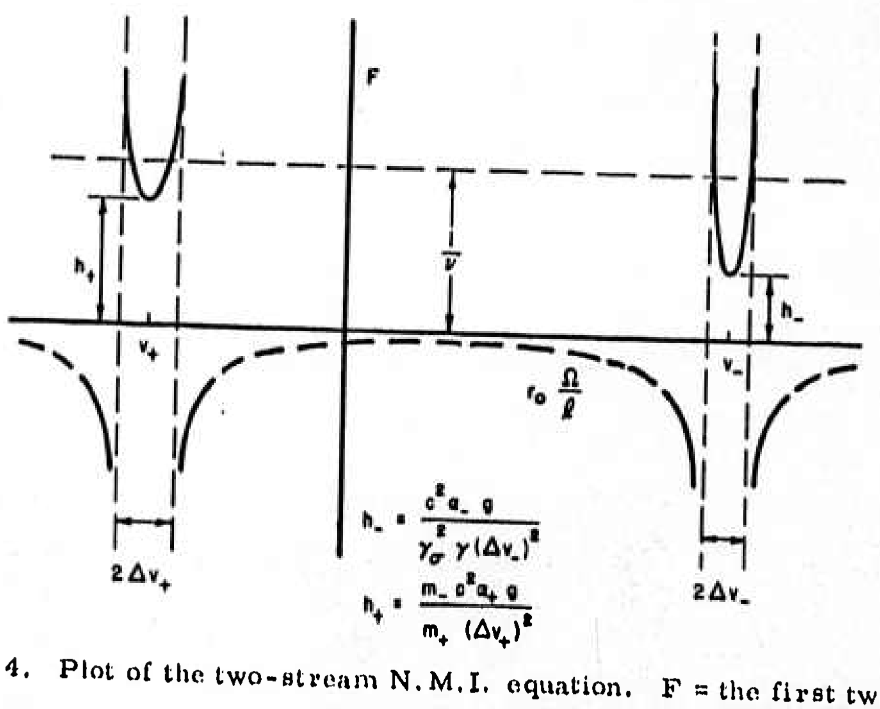

Fig. 4. Plot of the two-stream N.M.I. equation. F = the first two terms on the R.H.S. of eq. 3-20. The figure is drawn for a stable case and gives four real roots. Note that for a non-relativistic beam at one temperature $h_+ = h_-$. (see p 49)

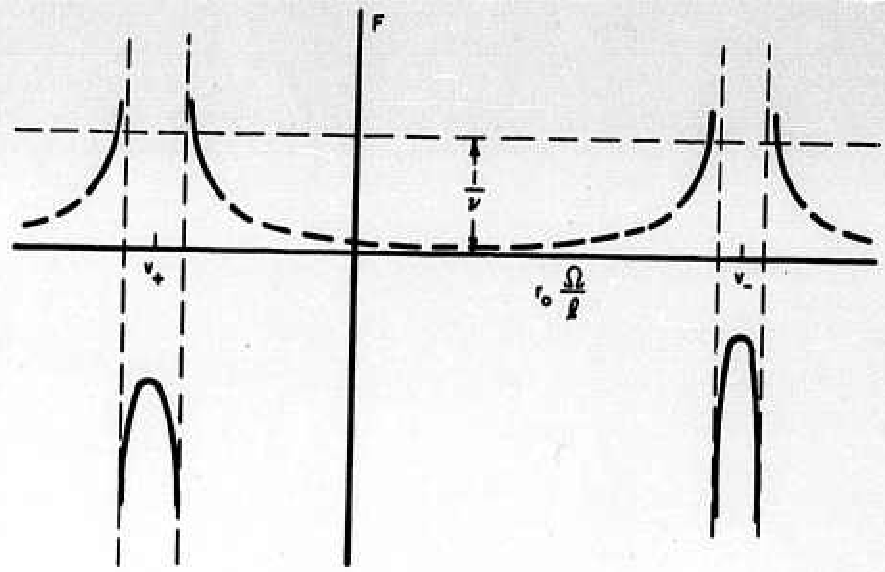

Fig. 5. Plot of the two stream longitudinal instability. F = the two terms on the R.H.S. of eq. 3-28. Note that if the numerator of the second term in this equation is negative, then the right side of the above figure is inverted about the horizontal axis. (see p 54)

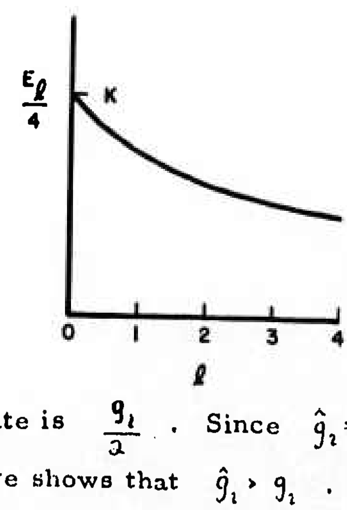

Fig. 6. The ordinate is $\frac{g_\ell}{2}$. Since $\hat{g}_\ell = \frac{g_{\ell-1} + g_{\ell+1}}{2}$, the concavity of the curve shows that $\hat{g}_\ell > g_\ell$. (see p 79)

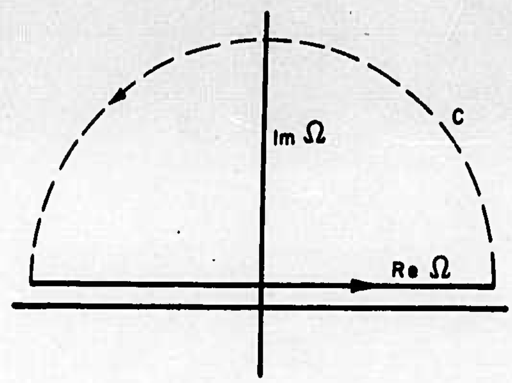

Fig. 7. The complex Ω plane used for the Nyquist diagram (see p 83)

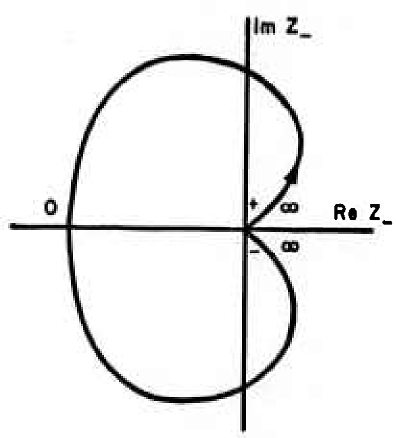

Fig. 8. Map of the curve C of Fig. 7 on the W plane for Maxwellian distributions. (see p 83)

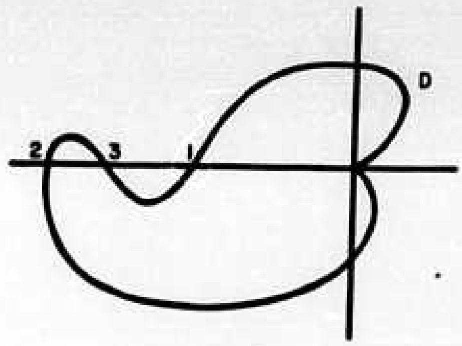

Fig. 9. Map of the curve C of Fig. 7 for some multi-peaked distribution function. (see p 85)

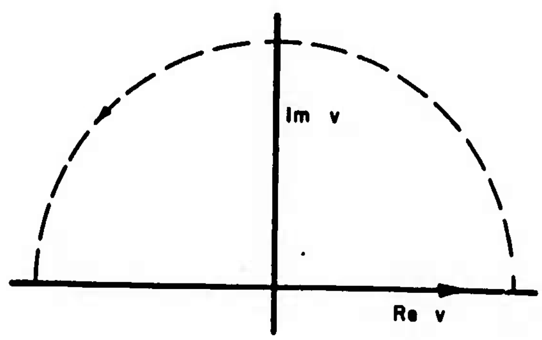

Fig. 10. The complex ν plane used for integrations involving a resonance shape distribution function. (see p 88)

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